Dr. Z’s Math151 Handout #3.10 [Derivatives of General Exponential and Logarithmic Functions]

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**Problem Type 3.10.1**: Differentiate the function \( f(x) = \text{Expression}(x) \), where the expression involves \( \ln x \) or \( \log_a x \).

**Example Problem 3.10.1**: Differentiate the function \( f(x) = x(\ln x)^{1/5} \).

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**Steps**

1. All the differentiation rules are still applicable! We just have a brand new function \( \ln x \) whose derivative is simply \( 1/x \). The derivative of \( \log_a x \) is \( 1/(x \ln a) \) (since \( \log_a x = \ln x / \ln a \)). But, before we differentiate \( \ln(\text{COMPLICATED}) \), you should use the \( \ln \) simplification rules to break the function into simple pieces [whenever possible, in this particular example, the argument of \( \ln \) is just \( x \), so there is nothing to simplify].

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**Example**

1. First we must use the product rule:

\[
 f'(x) = (x(\ln x)^{1/5})' = x'(\ln x)^{1/5} + x((\ln x)^{1/5})'.
\]

Using the chain rule for the second term, we get

\[
 1 \cdot (\ln x)^{1/5} + x \cdot (1/5) \cdot (\ln x)^{-4/5} \cdot (\ln x)'
\]

which equals

\[
  (\ln x)^{1/5} + \frac{x}{5} \cdot (\ln x)^{-4/5} \cdot (1/x) = (\ln x)^{1/5} + \frac{(\ln x)^{-4/5}}{5}.
\]
**Problem Type 3.10.2**: Use logarithmic differentiation to find the derivative of the function $f(x)$ which is a product of complicated powers and possibly exponentials.

**Example Problem 3.10.2**: Use logarithmic differentiation to find the derivative of the function $f(x) = (2x + 1)^6(x^3 - 4)^{11}$.

**Introduction.** For such functions you don’t need logarithmic differentiation but it helps! You use the fact that $(\ln f(x))' = \frac{f'(x)}{f(x)}$ hence $f'(x) = f(x)(\ln f(x))'$. The point is that in these cases it is much easier to differentiate $\ln f(x)$ (after simplification!), then $f(x)$.

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**Steps**

1. Using the rules of ln, simplify $\ln f(x)$ as much as possible.

   **Example**

   1. $\ln(2x + 1)^6(x^3 - 4)^{11} = \ln(2x + 1)^6 + \ln(x^3 - 4)^{11} = 6 \ln(2x + 1) + 11 \ln(x^3 - 4)$.

2. Differentiate each piece of the result of step 1.

   **Example**

   2. $(\ln f(x))' = 6(\ln(2x+1))' + 11(\ln(x^3-4))'$

   

   $\frac{6(2x + 1)'}{2x + 1} + \frac{11(x^3 - 4)'}{x^3 - 4} = \frac{12}{2x + 1} + \frac{33x^2}{x^3 - 4}$.

3. Multiply the answer from step 2 by $f(x)$ in order to get $f'(x)$.

   **Example**

   3. **Ans.**

   $f'(x) = \left(\frac{12}{2x + 1} + \frac{33x^2}{x^3 - 4}\right)(2x+1)^6(x^3-4)^{11}$.