Dr. Z's Math151 Handout #3.1

Problem Type 3.1.1: Compute the derivative at the fiven point $x = a$, of a given function $f(x)$, using the limit definition and find an equation of the tangent line.

Example Problem 3.1.1: Compute the derivative at $x = -2$, if

$$f(x) = \frac{1}{x+3} \quad ,$$

using the limit definition and find an equation of the tangent line.

Steps

1. Write down the general definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad ,$$

and implement the specific number a and the specific function $f(x)$ given in the problem.

2. Use algebra to simplify what's inside the limit, and then take the limit (with respect to h).

Example

1. Here $a = -2$ and $f(x) = \frac{1}{x+3}$. So

$$f'(-2) = \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} \quad ,$$

Using algebra, we get $f(-2+h) = \frac{1}{-2+h+3} = \frac{1}{h+1}$, and plugging-in we get $f(-2) = \frac{1}{-2+3} = 1$. So we have

$$f'(-2) = \lim_{h \to 0} \frac{\frac{1}{h+1} - 1}{h} \quad ,$$

2.

$$f'(-2) = \lim_{h \to 0} \frac{\frac{1}{h+1} - 1}{h} = \lim_{h \to 0} \frac{\frac{-h}{h+1}}{h} = \lim_{h \to 0} \frac{-1}{h+1}$$

$$= \frac{-1}{0+1} = -1 \quad .$$

So we finished the first part. $f'(-2) = -1$.

3. The **derivative** at $x = a$ is the **slope of the tangent-line** at the point $(a, f(a))$. Plug-in $x = a$ into the function to get its y-coordinate, and use the point-slope equation of a line $(y - y_0) = m(x - x_0)$.

3. $f(-2) = 1$, so our **point** is $(-2, 1)$, and we found out that the slope is $m = -1$. So the equation of the tangent-line is $(y - 1) = (-1)(x - (-2))$ that simplifies to $y = -x - 1$.

Ans.: The equation of the tangent line at the designated point is $y = -x - 1$.

Problem Type 3.1.2: The following limits represent a derivative $f'(a)$ for some function $f(x)$ and some number a. Find $f(x)$ and a.

First Version:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Or **Second Kind**:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example Problem 3.1.2: The following limits represent a derivative $f'(a)$ for some function $f(x)$ and some number a. Find $f(x)$ and a.

$$(a) \quad \lim_{h \to 0} \frac{(3+h)^4 - 81}{h}$$

$$(b) \quad \lim_{x \to \pi} \frac{\cos(x) - \cos(\pi)}{x - \pi}$$

Steps

Example

1. Decide which version is being used, and do some "parttern-recognition" to decipher, first a and then $f(x)$. The second version is more transparent, since the denominator of the limit is $x - a$, (and also the limit is at $x \to a$) so you get a right away, and the top is $f(x) - f(a)$ and it is immediate to get $f(x)$. If the first version is being used, then it a bit more tricky. The denominator does not give us any information, it is always h, but from the $f(a + h)$ part you should be able to get both f and a.

1. let's do (b) first, since it uses the second version. The bottom is $x - \pi$, so $a = \pi$ and the top is $\cos(x) - \cos(\pi)$, so $f(x) = \cos x$.

Ans. to (b): $f(x) = \cos x$, $a = \pi$.

Regarding (a), $(3+h)^4$ should correspond to $f(a + h)$, so $a = 3$ and $f(x) = x^4$.

Ans. to (a): $f(x) = x^4$, $a = 3$.