Dr. Z's Math151 Handout #2.8 [The Formal Definition of a Limit]

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Problem Type 2.8.1 : Prove rigorously that

$$\lim_{x \to c} f(x) = L \quad ,$$

for some numbers c and L and some function (usually very simple) function $f(x)$.

Example Problem 2.8.1: Prove rigorously that

$$\lim_{x \to 2} 3x + 1 = 7 \quad .$$

Steps

1. We have to show that for every $\epsilon > 0$ one can find a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad if \quad |x - c| < \delta \quad .$$

Spell it out for the given function $f(x)$, the place where the limit is taken, c, and the value of the limit L given by the problem. Then try to use algebra to bring both left-sides to be the same.

Example

1. Here $c = 2$, $L = 7$ and $f(x) = 3x + 1$. We have to show that for every $\epsilon > 0$ one can find a $\delta > 0$ such that

$$|3x + 1 - 7| < \epsilon \quad if \quad |x - 2| < \delta \quad .$$

Using simple algebra this is the same as

$$|3x - 6| < \epsilon \quad if \quad |x - 2| < \delta \quad .$$

$$3|x - 2| < \epsilon \quad if \quad |x - 2| < \delta \quad .$$

$$|x - 2| < \epsilon/3 \quad if \quad |x - 2| < \delta \quad .$$

2. If you are lucky (the problem is simpe enough) you would get the same left sides, and to conclude, you write down the expression in ϵ that emerged on the right side of the first $|x - c| < ....$

2. Equating both right hand sides, we can take $\delta = \epsilon/3$. In that case this becomes **manifesly true** (this is called a tautology in logic). "If A is true than A is true".

We have just proved that for **every** $\epsilon > 0$ one can **always** take $\delta = \epsilon/3$ such that

$$|f(x) - 7| < \epsilon \quad if \quad |x - 2| < \delta \quad .$$

Thus we have **rigoroulsy** proved indeed

$$\lim_{x \to 2} \; 3x + 1 = \; 7 \quad .$$

Warning: This topic is way over your head, and usually does not show up on the Final exam. Do not spend too much time on it. You do need however to do the above for functions as above of the form $ax + b$, since it may show up in Exam I.