Dr. Z's Math151 Handout #2.7 [The Intermediate Value Theorem]

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Problem Type 2.7.1 : If $f(x) = Expression(x)$, show that there is a number c such that $f(c) = Number$.

Example Problem 2.7.1: If $f(x) = x^3 - x^2 + 3x$, show that there is a number c such that $f(c) = 20$.

Steps	**Example**
1. Assuming that the function is continous (and usually it is), you have, by trial and error (or by plotting), find two numbers a and b such that $Expression(a) < Number$ and $Expression(b) > Number$.	**1.** Since, $f(x) = x^3 - x^2 + 3x$, $f(0) = 0$ and $f(10) = 930$ (there are many other possibilities, of course).
2. Since $Number$ is between $f(a)$ and $f(b)$, we know for sure that there is a number c such that $f(c) = 20$. In fact there must be at least one such c between a and b.	**2.** Since 20 is between 0 and 930, we know for sure that there is a number c such that $f(c) = 20$. In fact there must be at least one such c between 0 and 10.

Problem Type 2.7.2 : Find an interval of length $\frac{1}{N}$ in $[a, b]$ that contains a root of $f(x)$.

Example Problem 2.7.2: Find an interval of length $\frac{1}{4}$ in $[1, 2]$ that contains a root of $x^3 - 2x - 1$.

Steps

1. Plug-in
$x = a$, $x = a + (b - a)/N$,
$x = a + 2(b - a)/N$, $\ldots$,
$x = a + N(b - a)/N = b$, until two consecutive values have opposite signs.

Example

1. Here $f(x) = x^3 - 2x - 1$, so

$$f(1) = -2 \quad , \quad f(5/4) = -99/64,$$

$$f(3/2) = -5/8 \quad , \quad f(7/4) = 55/64 \quad .$$

2. Invoke the Intermediate Value Theorem to deduce that the function has a root (vanishes) in that first subinterval where it takes opposite signs, since one endpoint is positive while the other endpoint is negative, and since the function is continuous, it must assume the value 0 somewhere inside that subinterval.

2. Since $f(x) = x^3 - 2x - 1$ is negative at $x = 3/2$ and positive at $x = 7/4$, it follows by IVT that $f(x)$ has a root in the open interval $(\frac{3}{2}, \frac{7}{4})$.

A MidTerm I-style problem (#26 in "Review Problems for Midterm One", Spring 2008)

Exlain why $x^3 + x - 1$ has a real root in $(0, 1)$.

Solution: The function $f(x) = x^3+x-1$ is a continuous function (all polynomials are automatically continuous). At the left endpoint of our interval, $x = 0$, we have $f(0) = -1$, while at the right endpoint, we have $f(1) = 1$. Since at one endpoint the **value** of the function is **above** 0, while at the other endpoint the value is **below** 0, it follows by IVT that somewhere within our interval (we don't know exactly where) the function must be **exactly** 0. This is our root.