Dr. Z's Math151 Handout # 2.6 [Trigonometric Limits]

By Doron Zeilberger

Problem Type 2.6.1: Use the **Squeeze Theorem** to prove that

$$\lim_{x \to a} GoesToZero(x) \cdot Bounded(x) = 0$$

Example Problem 2.6.1 : Use the **Squeeze Theorem** to prove that

$$\lim_{x \to 0} \sqrt{x^3 + x^2}\, \sin(\pi/x) = 0$$

Steps

1. Show that

$$\lim_{x \to 0} GoesToZero(x)$$

indeed equals 0.

2. Show that $Bounded(x)$ is indeed bounded, at least near $x = a$.

Example

1.

$$\lim_{x \to 0} \sqrt{x^3 + x^2} = \sqrt{0^3 + 0^2} = \sqrt{0} = 0.$$

2. The sin function is always between -1 and 1 hence is always bounded. $\square$

Problem Type 2.6.2:

Evaluate

$$\lim_{t \to 0} ExpressionInSinesAndOrCosines(t)$$

Example Problem 2.6.2 : Evaluate

$$\lim_{x \to 0} \frac{\sin 7x}{3x} \quad .$$

Steps

1. The **official way** is to rearrange things in such a way that you would be able to use one (or both) of the following

$$\lim_{w \to 0} \frac{\sin w}{w} = 1 \quad .$$

$$\lim_{w \to 0} \frac{1 - \cos w}{w} = 0 \quad ,$$

where w is *whatever* that *goes to 0*.

Example

1. in order to accomodate the $\sin 7x$ we divide and multiply by $7x$:

$$\lim_{x \to 0} \frac{\sin 7x}{3x}$$

$$= \lim_{x \to 0} \frac{(\sin 7x)(7x)}{(7x)(3x)}$$

$$= \left(\lim_{x \to 0} \frac{\sin 7x}{7x} \right) \cdot \left(\lim_{x \to 0} \frac{7x}{3x} \right)$$

$$= \left(\lim_{7x \to 0} \frac{\sin 7x}{7x} \right) \frac{7}{3} =$$

$$= 1 \cdot \frac{7}{3} = \frac{7}{3} \quad .$$

Ans.: $\frac{7}{3}$.

Steps

2. The **unofficial** (shortcut) way is to replace $\sin w$ by w everywhere.

Example

2.

$$\lim_{x \to 0} \frac{\sin 7x}{3x}$$

$$= \lim_{x \to 0} \frac{7x}{3x} = \lim_{x \to 0} \frac{7}{3} = \frac{7}{3} \quad .$$

Problem Type 2.6.3:

Evaluate

$$\lim_{t\to 0} \ ExpressionInSinesAndOrCosines(t)$$

Example Problem 2.6.3 : Evaluate

$$\lim_{x\to 0} \frac{\cos 5x - \cos 3x}{x} \quad .$$

Steps

1. Use algebra to rearrange the top so that you would be able to take advantage of

$$\lim_{w\to 0} \frac{1-\cos w}{w} = 0 \quad ,$$

where w is *whatever* that *goes to 0*.

Example

1.

$$\lim_{x\to 0} \frac{\cos 5x - \cos 3x}{x}$$

$$= \lim_{x\to 0} \frac{(1-\cos 5x) - (1-\cos 3x)}{x}$$

$$= \lim_{x\to 0} \frac{(1-\cos 5x)}{x} - \lim_{x\to 0} \frac{(1-\cos 3x)}{x}$$

$$= 5 \lim_{x\to 0} \frac{(1-\cos 5x)}{5x} - 3 \lim_{x\to 0} \frac{(1-\cos 3x)}{3x}$$

$$= 5 \lim_{5x\to 0} \frac{(1-\cos 5x)}{5x} - 3 \lim_{3x\to 0} \frac{(1-\cos 3x)}{3x} = 0 - 0 = 0$$

Ans.: 0.

Note: Much later in this semester, you will learn a better way of doing these, called **L'Hôpital's rule**, but until then you have to do it today's way. In particular, in Midterm One you can't use L'Hôpital's rule, in case you are familiar with it.