Dr. Z’s Math151 Handout # 2.5 [Evaluating Limits Algebraically]

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**Problem Type 2.5.1**: Evaluate the limit if it exists:

\[ \lim_{x \to a} f(x) \]

**Example Problem 2.5.1**: Evaluate the limit if it exists:

\[ \lim_{x \to -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} \]

**Steps**

1. Try to plug \( x = a \) into the function, if it makes sense (i.e. the denominator is not zero), then the limit is that value. For example, \( \lim_{x \to -2} \frac{x^2 - 1}{x + 1} = \frac{(2^2 - 1)}{(2 + 1)} = 1 \). If the top is non-zero and the bottom 0, then it does not exists. If you get \( 0/0 \) then **simplify** as much as possible.

**Example**

1. Plugging in \( x = -4 \) in \( \frac{x^2 + 5x + 4}{x^2 + 3x - 4} \) gives 
   \( \frac{(-4)^2 + 5(-4) + 4}{(-4)^2 + 3(-4) - 4} \) which is \( 0/0 \). So we must **simplify**. Factoring the top and bottom we get
   \[ \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \frac{(x + 1)(x + 4)}{(x - 1)(x + 4)} = \frac{x + 1}{x - 1} \]

2. Try to plug in \( x = a \) again, if you still get \( 0/0 \), go back to step 1. Otherwise you are done.

**Example**

2. So

\[ \lim_{x \to -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \lim_{x \to -4} \frac{x + 1}{x - 1} \]

Plugging in \( x = -4 \),

\[ \frac{-4 + 1}{-4 - 1} = \frac{-3}{-5} = \frac{3}{5} \]

**Final Answer**: \( 3/5 \).
Problem Type 2.5.2: Evaluate the limit if it exists:

$$\lim_{x \to a} f(x)$$

where $f(x)$ features radical (square-root) signs.

Example Problem 2.5.2: Evaluate the limit if it exists:

$$\lim_{x \to 0} \frac{x}{\sqrt{2 + x} - \sqrt{2 - x}}$$

Steps

1. In a typical problem, you would have, either at the top or bottom, something of the form $\sqrt{SomeThing} - \sqrt{SomeThingElse}$. The trick is to multiply both top and bottom by the conjugate:

$$\sqrt{SomeThing} + \sqrt{SomeThingElse}$$

In other problems, you would have

$$\sqrt{SomeThing} + \sqrt{SomeThingElse}$$

in which case you would multiply both top and bottom by

$$\sqrt{SomeThing} - \sqrt{SomeThingElse}$$

Example

1. The conjugate of $\sqrt{2 + x} - \sqrt{2 - x}$ is $\sqrt{2 + x} + \sqrt{2 - x}$. Sticking it both at the top and bottom gives us:

$$\lim_{x \to 0} \frac{x}{\sqrt{2 + x} - \sqrt{2 - x}} = \lim_{x \to 0} \frac{x(\sqrt{2 + x} + \sqrt{2 - x})}{(\sqrt{2 + x} - \sqrt{2 - x})(\sqrt{2 + x} + \sqrt{2 - x})}$$
2. Use that famous rule \((A - B)(A + B) = A^2 - B^2\) to simplify

\((\sqrt{\text{SomeThing}} + \sqrt{\text{SomeThingElse}})\cdot(\sqrt{\text{SomeThing}} - \sqrt{\text{SomeThingElse}}) = \text{SomeThing} - \text{SomeThingElse}\).

Simplify further as much as you can.

\[
\begin{align*}
2. & = \lim_{x \to 0} \frac{x(\sqrt{2 + x} + \sqrt{2 - x})}{(\sqrt{2 + x} - \sqrt{2 - x})(\sqrt{2 + x} + \sqrt{2 - x})} \\
& = \lim_{x \to 0} \frac{x(\sqrt{2 + x} + \sqrt{2 - x})}{(\sqrt{2 + x})^2 - (\sqrt{2 - x})^2} \\
& = \lim_{x \to 0} \frac{x(\sqrt{2 + x} + \sqrt{2 - x})}{2x} \\
& = \lim_{x \to 0} \frac{\sqrt{2 + x} + \sqrt{2 - x}}{2} \\
& = \frac{\sqrt{2} + 0 + \sqrt{2} - 0}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}.
\end{align*}
\]

Ans.: \(\sqrt{2}\).

3. Plug-in \(x = a\).

\[
\begin{align*}
3. & = \frac{\sqrt{2 + 0} + \sqrt{2 - 0}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}.
\end{align*}
\]

Ans.: \(\sqrt{2}\).
A Midterm I style-problem  (from Review Problems for Midterm One, Spring 2008)

Evaluate the limit if it exists:

\[
\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3}
\]

Solution:

\[
\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3} = \lim_{x \to 3} \frac{(\sqrt{x+1} - 2)(\sqrt{x+1} + 2)}{(x - 3)(\sqrt{x+1} + 2)}
\]

\[
= \lim_{x \to 3} \frac{(\sqrt{x+1})^2 - 2^2}{(x - 3)(\sqrt{x+1} + 2)}
\]

\[
= \lim_{x \to 3} \frac{x + 1 - 4}{(x - 3)(\sqrt{x+1} + 2)}
\]

\[
= \lim_{x \to 3} \frac{x - 3}{(x - 3)(\sqrt{x+1} + 2)}
\]

\[
= \lim_{x \to 3} \frac{1}{\sqrt{x+1} + 2}
\]

\[
= \frac{1}{(\sqrt{3+1} + 2)} = \frac{1}{(\sqrt{4} + 2)} = \frac{1}{(2 + 2)} = \frac{1}{4} .
\]

Ans.: The limit exists and equals \(\frac{1}{4}\).