Problem Type 2.4.1: Explain why the function is discontinuous at $x = a$:

$$f(x) = \begin{cases} 
    \text{Expression}(x), & \text{if } x \neq a; \\
    \text{Number}, & \text{if } x = a.
\end{cases}$$

Example Problem 2.4.1: Explain why the function is discontinuous at $x = 1$:

$$f(x) = \begin{cases} 
    \frac{x^2 - x}{x^2 - 1}, & \text{if } x \neq 1; \\
    3, & \text{if } x = 1.
\end{cases}$$

Steps

1. Find

$$\lim_{x \to a} \text{Expression}(x),$$

let’s call it $b$.

Example

1. 

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}.$$ 

2. If the function $f(x)$ would have been continuous at $x = a$ then Number should have been the limit from part 1, what we called $b$. Since they are not the same, $f(x)$ is discontinuous at $x = a$.

2. If the function $f(x)$ would have been continuous at $x = 1$ then 3 should have been equal to 1/2. Since they are not the same, $f(x)$ is discontinuous at $x = 1$. 


Problem Type 2.4.2: If \( f(x) = Expression(x) \), show that there is a number \( c \) such that \( f(c) = Number \).

Example Problem 2.4.2: If \( f(x) = x^3 - x^2 + 3x \), show that there is a number \( c \) such that \( f(c) = 20 \).

Steps

1. Assuming that the function is continuous (and usually it is), you have, by trial and error (or by plotting), find two numbers \( a \) and \( b \) such that \( Expression(a) < Number \) and \( Expression(b) > Number \).

2. Since \( Number \) is between \( f(a) \) and \( f(b) \), we know for sure that there is a number \( c \) such that \( f(c) = 20 \). In fact there must be at least one such \( c \) between \( a \) and \( b \).

Example

1. Since \( f(x) = x^3 - x^2 + 3x \), \( f(0) = 0 \) and \( f(10) = 930 \) (there are many other possibilities, of course).

2. Since 20 is between 0 and 930, we know for sure that there is a number \( c \) such that \( f(c) = 20 \). In fact there must be at least one such \( c \) between 0 and 10.
Problem Type 2.4.3: Explain why the function is discontinuous at \( x = a \):

\[
f(x) = \begin{cases} 
\text{LeftExpression}(x), & \text{if } x < a; \\
\text{RightExpression}(x), & \text{if } x \geq a.
\end{cases}
\]

Example Problem 2.4.3: Explain why the function is discontinuous at \( x = 0 \):

\[
f(x) = \begin{cases} 
e^x, & \text{if } x < 0; \\
x^2, & \text{if } x \geq 0.
\end{cases}
\]

Steps

1. Find the limit from the left at \( x = a \),

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^-} \text{LeftExpression}(x),
\]

and the limit from the right there:

\[
\lim_{x \to a^+} f(x) = \lim_{x \to a^+} \text{RightExpression}(x).
\]

Example

1. In this problem, the limit from the left at \( x = 0 \) is:

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^x = e^0 = 1,
\]

and the limit from the right is

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0.
\]

2. If \( f(x) \) would have been continuous, then these two numbers should have been the same. Since they are not, the function \( f(x) \) is discontinuous at \( x = a \).

2. Since these two numbers (1 and 0) are not the same, the function is discontinuous at \( x = 0 \).
Problem from a past Final [Spring 2008, #3 (8 points)]: For what value of the constant $c$ is the function $f$ continuous for all $x$? Here

$$f(x) = \begin{cases} \, cx^2 + 3, & \text{if } x \geq 5; \\ \, cx - 3, & \text{if } x < 5. \end{cases}$$

Solution: The limit from the left at $x = 5$ is

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} cx - 3 = 5c - 3.$$  

The limit from the right at $x = 5$ is

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} cx^2 + 3 = c(5)^2 + 3 = 25c + 3.$$  

If the function is going to be continuous at $x = 5$, we must have that the limit from the left equals the limit from the right, so we must have:

$$5c - 3 = 25c + 3.$$  

Solving for $c$ we get: $20c = -6$, So $c = -3/10$.

Ans.: The value of $c$ that makes the function $f$ continuous for all $x$ is $c = -\frac{3}{10}$. **