Dr. Z’s Math151 Handout #4.4 [Indeterminate Forms and L’Hôpital’s Rule]

By Doron Zeilberger

Problem Type 4.4.1: Given certain limits of certain functions, \( f(x), g(x), \ldots \) at a designated point \( x = a \), determine whether the limits (at that very same point \( x = a \)) of the quotient \( f(x)/g(x) \), product \( f(x)g(x) \), difference \( f(x) - g(x) \), and exponentiation \( f(x)^{g(x)} \) are indeterminate limits.

Example Problem 4.4.1: Given that

\[
\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0, \\
\lim_{x \to a} p(x) = \infty, \quad \lim_{x \to a} q(x) = \infty,
\]

which of the following are indeterminate forms. (a) \( \lim_{x \to a} [f(x)/g(x)] \), (b) \( \lim_{x \to a} [f(x)p(x)] \), (c) \( \lim_{x \to a} [p(x) - q(x)] \). (d) \( \lim_{x \to a} [f(x)^{g(x)}] \).

Steps

Example

a. For a quotient the limit is indeterminate whenever ‘plugging in’ yields \( 0/0 \) or \( \infty/\infty \).

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0},
\]

hence this is an indeterminate form.

b. For a product the limit is indeterminate whenever ‘plugging in’ yields \( 0 \cdot \infty \) or \( \infty \cdot 0 \), (and of course \( 0 \cdot -\infty \) or \( -\infty \cdot 0 \)).

\[
\lim_{x \to a} [f(x)p(x)] = \\
\lim_{x \to a} f(x) \cdot \lim_{x \to a} p(x) = \\
0 \cdot \infty,
\]

hence the limit is indeterminate.

c. The limit of a difference is indeterminate whenever it is of the type \( \infty - \infty \) (or \( (-\infty) - (-\infty) \)).

\[
\lim_{x \to a} [p(x) - q(x)] = \infty - \infty,
\]

hence the limit is indeterminate.
d. The limit of an exponentiation is indeterminate whenever it is of the type 0^0, ∞^0, or 1^∞.

d. \[
\lim_{x \to a} [f(x)^{g(x)}] = 0^0
\]
hence the limit is indeterminate.
**Problem Type 4.4.2**: Use L’Hospital’s rule, if appropriate to evaluate

\[ \lim_{x \to a} \frac{TOP(x)}{BOT(x)} . \]

**Example Problem 4.4.2**: Use L’Hospital’s rule, if appropriate to evaluate

\[ \lim_{x \to 1} \frac{x^8 - 1}{x^3 - 1} . \]

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**Steps**

**Example**

1. First plug-in \( x = a \) into \( \frac{TOP(x)}{BOT(x)} \) and see whether \( \frac{TOP(a)}{BOT(a)} \) yields 0/0 or \( \infty / \infty \). If it does, then L’Hospital’s rule is applicable.

   1. Plugging-in \( x = 1 \) into \( \frac{x^8 - 1}{x^3 - 1} \) gives 0/0, so L’Hospital’s rule is applicable.

2. Invoke L’Hospital’s rule

   \[ \lim_{x \to a} \frac{TOP(x)}{BOT(x)} = \lim_{x \to a} \frac{TOP'(x)}{BOT'(x)} . \]

   If you still get an indeterminate form (in this example you don’t), keep doing it, until you get a doable limit.

   2. \[ \lim_{x \to 1} \frac{x^8 - 1}{x^3 - 1} = \lim_{x \to 1} \frac{(x^8 - 1)'}{(x^3 - 1)'} = \lim_{x \to 1} \frac{8x^7}{3x^2} . \]

3. Evaluate the limit, by simplifying and plugging-in.

   3. \[ \lim_{x \to 1} \frac{8x^5}{3} = \frac{8 \cdot 1^5}{3} = \frac{8}{3} . \]

**Ans.:** 8/3.
**Problem Type 4.4.3**: Same as 4.4.2, but now you have to use L'Hospital’s rule more than once

**Example Problem 4.4.3**: Use L'Hospital’s rule, if appropriate to evaluate

\[
\lim_{x \to 0} \frac{1 - \cos x}{x^2}.
\]

**Steps**

1. First plug-in \( x = a \) into \( \text{TOP}(x)/\text{BOT}(x) \) and see whether \( \text{TOP}(a)/\text{BOT}(a) \) yields 0/0 or \( \infty/\infty \). If it does, then L'Hospital’s rule is applicable.

   **Example**

   1. Plugging-in \( x = 0 \) into \( \frac{1 - \cos x}{x^2} \) gives 0/0, so L'Hospital's rule is applicable.

2. Invoke L'Hospital’s rule

   \[
   \lim_{x \to a} \frac{\text{TOP}(x)}{\text{BOT}(x)} = \lim_{x \to a} \frac{\text{TOP}'(x)}{\text{BOT}'(x)}.
   \]

   If you still get an indeterminate form (in this example you do!), keep doing it, until you get a doable limit.

   **Example**

   2. 

   \[
   \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{(1 - \cos x)'}{(x^2)'} = \lim_{x \to 0} \frac{\sin x}{2x}.
   \]

   Now plugging-in \( x = 0 \) still yields 0/0, so we have to do L'Hospital again.

   \[
   = \lim_{x \to 0} \frac{(\sin x)'}{(2x)'} = \lim_{x \to 0} \frac{\cos x}{2}.
   \]

3. Evaluate the limit, by simplifying (if necessary) and plugging-in.

   **Example**

   3. 

   \[
   = \frac{\cos 0}{2} = \frac{1}{2}.
   \]

   Ans.: 1/2.
Problem Type 4.4.4: Use L’Hospital’s rule (or any other method) to evaluate

\[ \lim_{x \to \infty} [Expression_1(x) - Expression_2(x)] \]

where one of the expressions is a radical (i.e. involves the square-root sign), and plugging-in gives \( \infty - \infty \).

Example Problem 4.4.4: Use L’Hospital’s rule, or any other method, to evaluate

\[ \lim_{x \to \infty} [\sqrt{x^2 + 3x} - x] \]

Steps

1. Multiply top and bottom by the conjugate \( Expression_1(x) + Expression_2(x) \), and simplify as much as you can, using \((a - b)(a + b) = a^2 - b^2\).

Example

\begin{align*}
1. \quad & \lim_{x \to \infty} [\sqrt{x^2 + 3x} - x] = \\
& \lim_{x \to \infty} \frac{(\sqrt{x^2 + 3x} - x)(\sqrt{x^2 + 3x} + x)}{\sqrt{x^2 + 3x} + x} = \\
& \lim_{x \to \infty} \frac{(\sqrt{x^2 + 3x})^2 - x^2}{\sqrt{x^2 + 3x} + x} = \\
& \lim_{x \to \infty} \frac{(x^2 + 3x) - x^2}{\sqrt{x^2 + 3x} + x} = \\
& \lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 3x} + x} .
\end{align*}

2. If you can get by without L’Hospital’s rule, don’t bother using it (it may be complicated). Try to use any other rules.

2. In this case you can use the ‘only the leading term’ counts as \( x \to \infty \), what I call ‘forget about the little ones’.

\[ = \lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + x}} \]

where we ignored \( 3x \) in view of the much more important \( x^2 \), and we get

\[ = \lim_{x \to \infty} \frac{3x}{x + x} = \lim_{x \to \infty} \frac{3}{2} = \frac{3}{2} \].

Ans.: 3/2.
**Problem Type 4.4.5**: Use L'Hospitall's rule (or any other method) to evaluate

\[
\lim_{x \to \infty} \frac{Expression_1(x)^{1/Expression_2(x)}}{Expression_2(x)}
\]

where plugging in will give \(\infty^0\).

**Example Problem 4.4.5**: Use L'Hospital's rule, or any other method, to evaluate

\[
\lim_{x \to \infty} \frac{x^{1/2x}}{x}
\]

**Steps**

1. Taking natural logarithms, evaluate instead

\[
\lim_{x \to \infty} \ln\left(\frac{x^{1/2x}}{x}\right) = \lim_{x \to \infty} \frac{\ln x}{2x}
\]

using L'Hospital's rule, if necessary.

2. But what you got now is not the answer but the log-natural of the answer. To get the answer to the problem, you have to undo the effect of \(\ln\) by exponentiating. So the final answer is \(\exp(Above\_Limit)\).

**Example**

1. \[
\lim_{x \to \infty} \ln\left(\frac{x^{1/2x}}{x}\right) = \lim_{x \to \infty} \frac{\ln x}{2x} = \lim_{x \to \infty} \frac{(\ln x)'}{(2x)'} = \lim_{x \to \infty} \frac{1/x}{2} = \lim_{x \to \infty} \frac{1}{2x} = 0.
\]

2. \textbf{Ans.}: \(e^0 = 1\).