Dr. Z's Math151 Handout #3.8 [Derivatives of Logarithmic Functions]

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Problem Type 3.8.1 : Differentiate the function $f(x) = Expression(x)$, where the expression involves $\ln x$ or $\log_a x$.

Example Problem 3.8.1: Differentiate the function $f(x) = x(\ln x)^{1/5}$.

Steps

1. All the differentiation rules are still applicable! We just have a brand new function $\ln x$ whose derivative is simply $1/x$. The derivative of $\log_a x$ is $1/(x \ln a)$ (since $\log_a x = \ln x / \ln a$). But, before we differentiate $\ln(COMPLICATED)$, you should use the ln simplification rules to break the function into simple pieces [whenever possible, in this particular example, the argument of ln is just x, so there is nothing to simplify].

Example

1. First we must use the product rule:

$$f'(x) = (x(\ln x)^{1/5})' = x'(\ln x)^{1/5} + x((\ln x)^{1/5})' \quad .$$

Using the chain rule for the second term,
we get

$$1 \cdot (\ln x)^{1/5} + x \cdot (1/5) \cdot (\ln x)^{-4/5} \cdot (\ln x)'$$

which equals

$$(\ln x)^{1/5} + (x/5) \cdot (\ln x)^{-4/5} \cdot (1/x) =$$

$$(\ln x)^{1/5} + \frac{(\ln x)^{-4/5}}{5} \quad .$$

Problem Type 3.8.2 : Use logarithmic differentiation to find the derivative of the function $f(x)$ which is a product of complicated powers and possibly exponentials.

Example Problem 3.8.2: Use logarithmic differentiation to find the derivative of the function $f(x) = (2x+1)^6(x^3-4)^{11}$.

Introduction. For such functions you don't need **logarithmic differentiation** but it helps! You use the fact that $(\ln f(x))' = \frac{f'(x)}{f(x)}$ hence $f'(x) = f(x)(\ln f(x))'$. The point is that in these cases it is much easier to differentiate $\ln f(x)$ (after simplification!), then $f(x)$.

Steps	**Example**
1. Using the rules of ln, simplify $\ln f(x)$ as much as possible.	**1.** $$\ln(2x+1)^6(x^3-4)^{11} = \ln(2x+1)^6 + \ln(x^3-4)^{11} = 6\ln(2x+1) + 11\ln(x^3-4) \quad .$$
2. Differentiate each piece of the result of step 1.	**2.** $$(\ln f(x))' = 6(\ln(2x+1))' + 11(\ln(x^3-4))' = 6\frac{(2x+1)'}{2x+1} + 11\frac{(x^3-4)'}{x^3-4} = \frac{12}{2x+1} + \frac{33x^2}{x^3-4} \quad .$$
3. Multiply the answer from step 2 by $f(x)$ in order to get $f'(x)$. **Do not Simplify!**.	**3. Ans.** $$f'(x) = \left(\frac{12}{2x+1} + \frac{33x^2}{x^3-4}\right)(2x+1)^6(x^3-4)^{11} \quad .$$