Dr. Z's Math151 Handout #3.7 [Higher Derivatives]

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Problem Type 3.7.1 : Find the first and second derivatives of the function $f(x) = Expression(x)$.

Example Problem 3.7.1: Find the first and second derivatives of the function $f(x) = \tan^{-1}(x^4)$.

Steps

1. These are really two differentiation problems. You first find the derivative (a.k.a. the first derivative), and then you take the derivative of the derivative. The second differentiation is usually (but not always!) more complicated. So let's take the first derivative.

2. Now take the second derivative, i.e. the derivative of the answer from the previous step. Use any differentiation rules that you need.

Example

1. By the chain rule,

$$\frac{d}{dx}\tan^{-1}(x^4) =$$

$$\frac{1}{1+(x^4)^2}\cdot\frac{d}{dx}(x^4) =$$

$$\frac{4x^3}{1+x^8} \quad .$$

2. Using the quotient rule, we get

$$\frac{d^2}{dx^2}\tan^{-1}(x^4) =$$

$$\frac{d}{dx}\frac{4x^3}{1+x^8} =$$

$$\frac{(4x^3)'(1+x^8)-(4x^3)(1+x^8)'}{(1+x^8)^2} =$$

$$\frac{(12x^2)\cdot(1+x^8)-(4x^3)\cdot(8x^7)}{(1+x^8)^2} =$$

$$\frac{12x^2+12x^{10}-32x^{10}}{(1+x^8)^2} =$$

$$\frac{12x^2-20x^{10}}{(1+x^8)^2} =$$

$$\frac{4x^2(3-5x^8)}{(1+x^8)^2} \quad .$$

Problem Type 3.7.2 : An equation of motion is given, where s is in unit of distance and t is in units of time. Find (a) The times at which the acceleration is 0. (b) the displacement and velocity at these times.

Example Problem 3.7.2: As above with meters, second, and $s = t^4 - 4t^3 + 2$.

Steps	**Example**
1. Find the velocity $v = ds/st$ and the acceleration $a = dv/dt = d^2s/dt^2$.	**1.** $$v = \frac{d}{dt}(t^4 - 4t^3 + 2) = 4t^3 - 12t^2 \quad .$$ $$a = \frac{dv}{dt} = \frac{d}{dt}(4t^3 - 12t^2) = 12t^2 - 24t \quad .$$
2. Set the acceleration a equal to 0, and solve for t.	**2.** $12t^2 - 24t = 0$ hence $12t(t-2) = 0$, whose solutions are $t = 0$ and $t = 2$.
3. Plug-in the value(s) of t found in step 2 into s of the problem, and v that you found in step 1.	**3.** At $t = 0$: $$s(0) = 0^4 - 4 \cdot 0^3 + 2 = 2$$ $$v(0) = 4 \cdot 0^3 - 12 \cdot 0^2 = 0 \quad .$$ At $t = 2$: $$s(2) = 2^4 - 4 \cdot 2^3 + 2 = 16 - 32 + 2 = -14 \quad ,$$ $$v(2) = 4 \cdot 2^3 - 12 \cdot 2^2 = 32 - 48 = -16 \quad .$$.

Problem Type 3.7.3 : For what values of r does the function $y = e^{rx}$ satisfy the equation $ay'' + by' + cy = 0$. (a, b, c are some specific numbers).

Example Problem 3.7.3: For what values of r does the function $y = e^{rx}$ satisfy the equation $2y'' + y' - y = 0$.

Steps

1. Using the facts that if $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2e^{rx}$ plug these into $ay'' + by' + cy = 0$ getting

$$(ar^2 + br + c)e^{rx} = 0 \quad .$$

2. Solve for r the equation

$$(ar^2 + br + c)e^{rx} = 0 \quad .$$

Since e^{rx} is never 0, this is the same as solving

$$ar^2 + br + c = 0 \quad ,$$

a quadratic equation in r.

Example

1. Using the facts that if $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2e^{rx}$ plug these into $2y'' + y' - y = 0$ getting

$$(2r^2 + r - 1)e^{rx} = 0 \quad .$$

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2. We have to solve

$$2r^2 + r - 1 = 0 \quad ,$$

which by factorizing, is the same as

$$(2r - 1)(r + 1) = 0 \quad ,$$

so the roots are

Ans.: $r = 1/2$ and $r = -1$.