Dr. Z's Math151 Handout #3.6 [Implicit Differentiation]

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Problem Type 3.6.1 : Find dy/dx by implicit differentiation, where you know that $Expression_1(x, y) = Expression_2(x, y)$.

Example Problem 3.6.1: Find dy/dx by implicit differentiation if $x^2 - 2xy + y^3 = 5x$

Steps

1. Make the right side zero by moving everything to the left side. Getting something of the form $F(x, y) = 0$.

2. Use all the rules of differentiation treating y as an abstract function of x, and using the chain rule. So, for example, while $(x^3)' = 3x^2$, $(y^3)' = 3y^2y'$. So $(y^n)' = ny^{n-1}y'$, $(\sin y)' = (\cos y)y'$, $(e^y)' = e^y y'$ etc. In general $(f(y))' = f'(y)y'$, so you can pretend that y is a variable, but whenever you differentiate an expression in y, stick a y' at the end. You also may use the product and (sometimes) the quotient rule.

3. Use algebra to solve for y' in terms of x and y, by keeping all the terms involving y' on the left, and moving all the other terms to the right. Then factor y' out at the left, and finally solve for y'.

Example

1. $x^2 - 2xy + y^3 - 5x = 0$

2.

$$(x^2 - 2xy + y^3 - 5x)' = (x^2)' - (2xy)' + (y^3)' - (5x)'$$

$$= 2x - ((2x)'y + 2xy') + 3y^2y' - 5 =$$

$$2x - (2y + 2xy') + 3y^2y' - 5 = 2x - 2y - 2xy' + 3y^2y' - 5 = 0$$

3. $-2xy' + 3y^2y' = 2y - 2x + 5$ hence $(3y^2 - 2x)y' = 2y - 2x + 5$ and we get **Ans.**: $y' = \frac{2y-2x+5}{3y^2-2x}$.

Problem Type 3.6.2 : Use implicit differentiation to find the equation of the tangent line to the curve $F(x, y)$ at the point (a, b).

Example Problem 3.6.2: Use implicit differentiation to find the equation of the tangent line to the curve $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at the point $(0, \frac{1}{2})$.

Steps

1. First (optionally) you may want to make sure that the given point indeed lies on the curve by plugging it in. Then perform implicit differentiation.

Example

1. $0^2 + (1/2)^2 = (2 \cdot 0^2 + 2 \cdot (1/2)^2 - 0)^2$, so the point is OK. Now

$$(x^2 + y^2)' - ((2x^2 + 2y^2 - x)^2)' =$$

$$2x+2yy'-2(2x^2+2y^2-x)(2x^2+2y^2-x)' =$$

$$2x+2yy'-2(2x^2+2y^2-x)(4x+4yy'-1) = 0 \quad .$$

2. Now do not solve for y' right away (like you did in the 3.6.1), but you plug-in for (x, y) the given point (a, b). Then you solve for y' getting a *number*, which is the slope at the given point.

2. Plugging in $x = 0$, $y = \frac{1}{2}$ gives

$$2{\cdot}0+2(1/2)y'-2(2{\cdot}0^2+2{\cdot}(1/2)^2-0)(4{\cdot}0+4(1/2)y'-1) = 0 \quad .$$

Hence $y' - (2y' - 1) = 0$ and $-y' + 1 = 0$, so, solving for y' we get $y' = 1$, i.e. the slope of the tangent, m, is 1.

3. As usual the equation of the tangent is $(y - b) = m(x - a)$

3. $y - \frac{1}{2} = 1(x - 0)$. **Ans.:** $y = x + \frac{1}{2}$.

Problem Type 3.6.3 : Find the derivative of an expression involving one of the inverse trig functions $\sin^{-1}$, $\cos^{-1}$,$\tan^{-1}$, etc.

Example Problem 3.6.3: Find the derivative of $y = \cos^{-1}(3x+1)$.

Steps

1. Use all applicable differentiation rules, remembering the derivatives of the new functions:

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \quad ,$$

$$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}} \quad ,$$

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2} \quad .$$

2. Simplify

Example

1. By the chain rule

$$\frac{d}{dx}(\cos^{-1}(3x+1)) = -\frac{1}{\sqrt{1-(3x+1)^2}} \cdot \frac{d}{dx}(3x+1)$$

2.

$$-\frac{3}{\sqrt{1-(9x^2+6x+1)}} =$$

$$-\frac{3}{\sqrt{-9x^2-6x}} =$$

$$-\frac{3}{\sqrt{-3x(3x+2)}} \quad .$$