Dr. Z's Math151 Handout #3.4 [Derivatives of Trigonometric Functions]

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**Problem Type 3.4.1**: Differentiate an expression involving products and/or quotients of expressions containing *trig* functions.

**Example Problem 3.4.1**: Differentiate

\[ y = \frac{1 + \cos x}{x + \sin x}. \]

**Steps**

1. Use the product or quotient rule to differentiate the given expression, remembering that \((\sin x)' = \cos x, \ (\cos x)' = -\sin x, \ (\tan x)' = \sec^2 x\). You don’t really need to remember the more obscure formulas \((\csc x)' = -\csc x \cot x, \ (\sec x)' = \sec x \tan x, \ (\cot x)' = -\csc^2 x\), since you can always replace \(\csc x\) by \(1/\sin x\), etc.

2. Use algebra, and possibly trig identities, the most important one being

\[ \sin^2 x + \cos^2 x = 1, \]

to simplify.

**Example**

1. \[
\begin{align*}
& (1 + \cos x)' = \frac{1 + \cos x}{x + \sin x} \\
& (1 + \cos x)'(x + \sin x) - (1 + \cos x)(x + \sin x)' = \frac{(1 + \cos x)(x + \sin x)'}{(x + \sin x)^2} = \frac{(-\sin x)(x + \sin x) - (1 + \cos x)(1 + \cos x)}{(x + \sin x)^2}.
\end{align*}
\]

2. \[
\begin{align*}
& -(x \sin x - \sin^2 x) - (1 + 2 \cos x + \cos^2 x) = \frac{(1 + 2 \cos x + \cos^2 x)}{(x + \sin x)^2} = \\
& -1 - x \sin x - 2 \cos x - (\sin^2 x + \cos^2 x) = \frac{-1 - x \sin x - 2 \cos x - (1)}{(x + \sin x)^2} = \\
& -2 - x \sin x - 2 \cos x = \frac{-2 - x \sin x - 2 \cos x}{(x + \sin x)^2}.
\end{align*}
\]

**Ans.** \[ -\frac{2 - x \sin x - 2 \cos x}{(x + \sin x)^2}. \]
Problem Type 3.4.2: Find limits of expressions involving trig. functions. (usually at 0).

Example Problem 3.4.2: Find the limit

$$
\lim_{x \to 0} \frac{\sin 7x}{\sin 5x}
$$

Steps

1. **Important Fact:** When you are taking a limit as \( x \to 0 \), and the argument of \( \sin \) is 0 when \( x = 0 \), then you can ‘forget about the \( \sin \)’, meaning replacing \( \sin(\text{Whatever}) \) by \( \text{Whatever} \).

Example

1.

\[
\lim_{x \to 0} \frac{\sin 7x}{\sin 5x} = \\
\lim_{x \to 0} \frac{7x}{5x} = \\
\lim_{x \to 0} \frac{7}{5} = \\
\frac{7}{5}.
\]