Problem Type 3.11.1: Use the Linear Approximation to estimate \( \Delta f = f(a + h) - f(a) \) for the given function \( f(x) \), for the given \( a \) and \( h \).

Example Problem 3.11.1: Use the Linear Approximation to estimate \( \Delta f = f(4 + 0.01) - f(4) \) for \( f(x) = \frac{1}{x} \).

Steps

1. Compute \( f'(x) \), and decide who is \( a \) and who is \( h \).
2. Set up the formula

\[
\Delta f \approx f'(a) h ,
\]

and do the plugging-in

\[
\text{Ans.: } \Delta f \approx \frac{-1}{1600}.
\]

Problem Type 3.11.2: Estimate the quantity using the Linear Approximation.

Example Problem 3.11.2: Estimate the quantity

\[
\frac{1}{\sqrt{97}} - \frac{1}{10}.
\]

Steps

1. By “pattern-recognition” decide (i) what is the function (ii) what is the “nice” point \( a \) (iii) what is the deviation \( h \).

Also find \( f'(x) \).

Example

1. Here \( f(x) = \frac{1}{\sqrt{x}} = x^{-1/2} \). The “nice” point near 97 is \( a = 100 \) and \( h = -3 \).

\[
f'(x) = (-1/2)x^{-3/2} = \frac{-1}{2\sqrt{x^3}}.
\]
2. Set up the formula
\[
\Delta f \approx f'(a)h ,
\]
and implement it for the specific problem.

\[
\Delta f \approx f'(100)(-3) = \frac{-1}{2(\sqrt{100})^3}(-3) = \frac{3}{2000} .
\]

**Problem Type 3.11.3:** Find the linearization at \(x = a, y = \text{SomeFunction}(x), a = \text{SomeNumber}.\)

**Example Problem 3.11.3:** Find the linearization at \(x = a, y = (5 + x^2)^{-1/2}, a = 2.\)

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**Steps**

1. Find \(f'(x).\)

\[
f'(x) = ((5 + x^2)^{-1/2})' = (-1/2)(5 + x^2)^{-3/2}(5 + x^2)' =
\]
\[
= (-1/2)(5 + x^2)^{-3/2}(2x) = -x(5 + x^2)^{-3/2} = \frac{-x}{(\sqrt{5 + x^2})^3}
\]

2. Plug everything into the formula

\[
L(x) = f'(a)(x-a) + f(a) ,
\]

\[
L(x) = f'(a)(x-a)+f(a) = \frac{-2}{(\sqrt{5 + 2^2})^3}(x-2)+(5+2^2)^{-1/2}
\]
\[
= \frac{-2}{2^7}(x-2)+(9)^{-1/2} = \frac{-2}{27}(x-2)+\frac{1}{3} .
\]

**Ans.:** \(L(x) = \frac{-2}{27}(x-2)+\frac{1}{3} .\)
Problem from a Previous Final (Spring 2008, #4 (9 points)).

Let \( f(x) = \sqrt{1 - x} \)

(a) (6 points) Using the linear approximation of \( f(x) \) at \( a = -3 \) compute an approximation to \( f(-4) \).

(b) (3 points) Use \( f'' \) (concavity) to determine whether your approximation is larger or smaller than the true value of \( f(-4) \).

Solution

(a) \( f(x) = \sqrt{1 - x} = (1 - x)^{1/2} \). Since \( a = -3 \), \( f(-3) = 4^{1/2} = 2 \). We also have

\[
f'(x) = (1/2)(1 - x)^{-1/2}(-1) = \frac{-1}{2\sqrt{1 - x}}.
\]

Plugging-in \( x = -3 \) gives

\[
f'(-3) = \frac{-1}{2\sqrt{1 - (-3)}} = \frac{-1}{2\sqrt{4}} = \frac{-1}{4}.
\]

The linear approximation is

\[
L(x) = f'(-3)(x - (-3)) + f(-3) = \frac{-(x + 3)}{4} + 2.
\]

And when \( x = -4 \), we get

\[
L(-4) = \frac{-(4 + 3)}{4} + 2 = \frac{9}{4}.
\]

Ans. to (a): The approximation to \( f(-4) \) using the Linear approximation is \( \frac{9}{4} = 2.25 \).

(b) \( f''(x) = (-1/4)(1 - x)^{-3/2} \). So \( f''(-4) = (-1/4) \cdot 5^{3/2} \) is negative, this means that the approximation is larger than the true value of \( f(-4) \).