Problem Type 3.10.1: If $F(x, y) = c$ and $dy/dt = a$, find $dx/dt$ when $y = b$.

Example Problem 3.10.1: If $x^3 + y^3 = 9$ and $dy/dt = 6$ find $dx/dt$ when $y = 2$.

Steps

1. Find the corresponding value of $x$ by solving $F(x, b) = c$.

Example

1. When $y = 2$, $x^3 + y^3 = 9$ becomes $x^3 + 2^3 = 9$ so $x = 1$.

2. Differentiate the relation $F(x, y) = c$ with respect to $t$, like in implicit differentiation, treating $x$ and $y$ as unknown functions of $t$.

Example

2. 

\[
\frac{d}{dt} (x^3 + y^3) = 0 ,
\]

so

\[
3x^2 \frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 0 .
\]

3. In the relationship that you got in step 2 between $x, t, dx/dt, dy/dt$, plug-in the values for $x, y$, and $dy/dt$ and solve for $dx/dt$.

Example

3. 

\[
3 \cdot 1^2 \frac{dx}{dt} + 3 \cdot 2^2 \cdot 6 = 0 ,
\]

so

\[
\text{Ans.}: \frac{dx}{dt} = -24 .
\]

Problem Type 3.10.2: A water trough is of length $L$ and a cross-section has the shape of an isosceles trapezoid that has width $a$ at the bottom, $b$ at the top, and height $H$. If the trough is being filled with water at a rate of $d$, how fast is the water level rising when the water had depth $x$?

Example Problem 3.10.2:

A water trough is of length 30 meters and a cross-section has the shape of an isosceles trapezoid that has width .3 meters at the bottom, .8 meters at the top, and height .5 meters. If the trough
is being filled with water at a rate of \(0.2 \text{ m}^3/\text{min}\), how fast is the water level rising when the water had depth \(0.3\) meters?
Steps

1. If the water-height is $x$, by similar triangles the top side of the water trapezoid is
   
   \[
   a + \frac{(b - a)x}{H}
   \]
   
   since when $x = 0$ it is $a$ and when $x = H$ it is $b$. So the area of the water cross-section is
   
   \[
   \frac{x}{2}(2a + \frac{(b - a)x}{H})
   \]
   
   and multiplying by the length, the volume is
   
   \[
   V(x) = \frac{Lx}{2}(2a + \frac{(b - a)x}{H})
   \]
   
   this is your expression for the volume.

2. Find $dV/dt$ by implicit differentiation, getting an expression featuring $x$ and $dx/dt$.

3. Plug-in the known values of $x$ and $dV/dt$ and solve for $dx/dt$.

Example

1. If the water-height is $x$, by similar triangles the top side of the water trapezoid is
   
   \[
   .3 + \frac{(.8 - .3)x}{.5} = .3 + x
   \]
   
   since when $x = 0$ it is .3 and when $x = .5$ it is .8. So the area of the water cross-section is
   
   \[
   \frac{x}{2}(.3 + (.3 + x)) = \frac{x}{2}(.6 + x)
   \]
   
   and multiplying by the length, the volume is
   
   \[
   V(x) = \frac{30x}{2}(.6 + x) = 15x(.6 + x) = 9x + 15x^2
   \]
   
   this is your expression for the volume.

2. 

   \[
   \frac{dV}{dt} = \frac{d}{dt}(9x + 15x^2) = 9\frac{dx}{dt} + 30x\frac{dx}{dt}
   \]

3. 

   \[
   .2 = 9\frac{dx}{dt} + 30 \cdot .3\frac{dx}{dt}
   \]
   
   so
   
   \[
   .2 = 18\frac{dx}{dt}
   \]
   
   and

   Ans.: \[
   \frac{dx}{dt} = \frac{.2}{18} = \frac{1}{90} = .0111 \text{ m/min}
   \]
Problem Type 3.10.3: If two resistors with resistance $R_1$ and $R_2$ are connected in parallel, then the total resistance is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$ 

If $R_1$ and $R_2$ are increasing at a rate of $a$ and $b$ ohms per sec. respectively, how fast is $R$ changing when $R_1 = A$ and $R_2 = B$.

Example Problem 3.10.3: If two resistors with resistance $R_1$ and $R_2$ are connected in parallel, then the total resistance is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$ 

If $R_1$ and $R_2$ are increasing at a rate of $.6$ and $.4$ ohms per sec. respectively, how fast is $R$ changing when $R_1 = 160$ and $R_2 = 200$.

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**Steps**

1. Differentiate the above relation with respect to $t$, getting

$$\frac{d}{dt} \frac{1}{R} = \frac{d}{dt} \frac{1}{R_1} + \frac{d}{dt} \frac{1}{R_2}.$$ 

So

$$\frac{-dR}{R^2} = \frac{-dR_1}{R_1^2} + \frac{-dR_2}{R_2^2}.$$

2. Find the value of $R$ corresponding to $R_1 = A$ and $R_2 = B$ by solving for $R$ the equation $1/R = 1/R_1 + 1/R_2$. 

Example

1. Same as in general.

2. Find the value of $R$ corresponding to $R_1 = 160$ and $R_2 = 200$ by solving for $R$ the equation $1/R = 1/160 + 1/200 = 9/800$, so $R = 800/9$. 

3. Plug the values for $R$ found in step 2, and the values for $R_1, R_2 \frac{dR_1}{dt}, \frac{dR_2}{dt}$ given by the problem into the relationship found in step 1, and solve for $\frac{dR}{dt}$,

$$\frac{-\frac{dR}{dt}}{(800/9)^2} = \frac{-0.6}{160^2} + \frac{-0.4}{200^2}$$

So

$$\frac{dR}{dt} = (800/9)^2 \left( \frac{0.6}{160^2} + \frac{0.4}{200^2} \right)$$

$$= (0.6)(5/9)^2 + (0.4)(4/9)^2 =$$

$$= (15+6.4)/81 = 21.4/81 = 107/405 \text{ ohm/s.}$$