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**Problem Type 2.4.1:**

Use the given graph of \( f(x) \) to find a number \( \delta \) such that \( |f(x) - f(a)| < \epsilon \) whenever \( |x - a| < \delta \).

(A graph is given with dashed lines at \( x = a \) and \( y = f(a) \), and red lines at \( y = f(a) \pm \epsilon \), and \( x = ? \) continued down. [see ex. 6, sect. 2.4, p. 122, for an example].)

**Example Problem 2.4.1:** Ex. 6, sect. 2.4, p. 122. Use the given graph \( f(x) = x^2 \) to find a number \( \delta \) such that

\[
|x^2 - 1| < \frac{1}{2} \quad \text{whenever} \quad |x - 1| < \delta.
\]

[Refer to the book for the diagram].

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**Steps**

1. Find the values of the question marks on the \( x \)-axis. In other words, solve \( f(x) = f(a) - \epsilon \) and \( f(x) = f(a) + \epsilon \), let’s call them \( x_1 \) and \( x_2 \). [Note that this only works if the function goes up, or goes down in the given interval].

2. Take \( \delta \) to be the smaller of the two numbers \( |x_1 - a| \) and \( |x_2 - a| \).

**Example**

1. The solution of

\[
x^2 = 1 - 1/2
\]

is

\[
x = 1/\sqrt{2} = .707\ldots
\]

The solution of

\[
x^2 = 1 + 1/2
\]

is

\[
x = \sqrt{3/2} = 1.224\ldots
\]

2. \( |.707\ldots - 1| = |-.292\ldots| = .292 \)

\( |1.224\ldots - 1| = |.224| = .224 \).

Hence \( \delta = .224 \), the smallest of these two numbers.
Problem Type 2.4.2: Prove the statement using the $\epsilon, \delta$ definition of limit.

$$\lim_{x \to a} f(x) = A .$$

Example Problem 2.4.2: Prove the statement using the $\epsilon, \delta$ definition of limit.

$$\lim_{x \to 4} \frac{x}{2} = 2 .$$

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Steps

1. ‘Guess’ the value for $\delta$ (as an expression in $\epsilon$), by manipulating

   $$|f(x) - A| < \epsilon ,$$

   and trying to make it look like

   $$|x - a| < \text{something} .$$

   The resulting ‘something’ (that depends on $\epsilon$) is your ‘guessed’ $\delta$.

2. Using the ‘guessed’ $\delta$ prove that

   $$|x - a| < \delta \quad \text{implies} \quad |f(x) - A| < \epsilon .$$

Example

1. 

   $$|\frac{x}{2} - 2| < \epsilon$$

   is equivalent to

   $$|x - 4| < 2\epsilon ,$$

   so the ‘something’ is $2\epsilon$. Hence the ‘guessed’ $\delta$ is $2\epsilon$.

2. Dividing the inequality

   $$|x - 2| < 2\epsilon$$

   by 2 yields

   $$|\frac{x}{2} - 2| < \epsilon ,$$

   which is the desired conclusion.