Dr. Z's Math151 Handout #2.4 [The Precise Definition of a Limit]

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Problem Type 2.4.1 :

Use the given graph of $f(x)$ to find a number δ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

(A graph is given with dashed lines at $x = a$ and $y = f(a)$, and red lines at $y = f(a) \pm \epsilon$, and $x =?$ continued down. [see ex. 6, sect. 2.4, p. 122, for an example].)

Example Problem 2.4.1: Ex. 6, sect. 2.4, p. 122. Use the given graph $f(x) = x^2$ to find a number δ such that

$$|x^2 - 1| < \frac{1}{2} \quad \textit{whenever} \quad |x - 1| < \delta \quad .$$

[Refer to the book for the diagram].

Steps

1. Find the values of the question marks on the x-axis. In other words, solve $f(x) = f(a)-\epsilon$ and $f(x) = f(a)+\epsilon$, let's call them x_1 and x_2. [Note that this only works if the function goes up, or goes down in the given interval].

2. Take δ to be the smaller of the two numbers $|x_1 - a|$ and $|x_2 - a|$.

Example

1. The solution of

$$x^2 = 1 - 1/2$$

is

$$x = 1/\sqrt{2} = .707\ldots$$

The solution of

$$x^2 = 1 + 1/2$$

is

$$x = \sqrt{3/2} = 1.224\ldots$$

2. $|.707\ldots - 1| = | - .292\ldots| = .292$
$|1.224\ldots - 1| = |.224| = .224$.

Hence $\delta = .224$, the smallest of these two numbers.

Problem Type 2.4.2 : Prove the statement using the ϵ, δ definition of limit.

$$\lim_{x \to a} f(x) = A \quad .$$

Example Problem 2.4.2: Prove the statement using the ϵ, δ definition of limit.

$$\lim_{x \to 4} \frac{x}{2} = 2 \quad .$$

Steps	**Example**
1. 'Guess' the value for δ (as an expression in ϵ), by manipulating $$\lvert f(x) - A \rvert < \epsilon \quad ,$$ and trying to make it look like $$\lvert x - a \rvert < \mathit{something} \quad .$$ The resulting 'something' (that depends on ϵ) is your 'guessed' δ.	**1.** $$\lvert \frac{x}{2} - 2 \rvert < \epsilon$$ is equivalent to $$\lvert x - 4 \rvert < 2\epsilon \quad ,$$ so the 'something' is 2ϵ. Hence the 'guessed' δ is 2ϵ.
2. Using the 'guessed' δ prove that $$\lvert x - a \rvert < \delta \quad \mathit{implies} \quad \lvert f(x) - A \rvert < \epsilon \quad .$$	**2.** Dividing the inequality $$\lvert x - 2 \rvert < 2\epsilon$$ by 2 yields $$\lvert \frac{x}{2} - 2 \rvert < \epsilon \quad ,$$ which is the desired conclusion.