Dr. Z’s Math151 Handout # 2.2 [The Limit of a Function]

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Problem Type 2.2.1: Given a graph of a function, find the limits from the left, limit from the right, limit (if it exists), at various points, as well as some function values.

Example Problem 2.2.1: Ex. 7 of section 2.2 in Stewart’s book (p. 102).

Steps

Limit from the Left: If the function ‘looks’ continuous from the left, at the given point, then the limit from the left is the natural ‘continuation’ (possibly indicated by a hollow dot) from the left.

Limit from the Right: If the function ‘looks’ continuous from the right, at the given point, then the limit from the right is the natural ‘continuation’ (possibly indicated by a hollow dot) from the right.

Limit: If BOTH the ‘limit from the left’ \( \lim_{x \to a^-} \) and the ‘limit from the right’ \( \lim_{x \to a^+} \) exist, AND they are equal, then the limit, \( \lim_{x \to a} \), exists, and is equal to their common value. Otherwise NOT!

Function Value: To find \( f(a) \) from the curve, if the curve passes ‘smoothly’ through \( x = a \), then \( f(a) \) has its obvious value. Otherwise it is the value of the filled-in dot.

Example

(a) \(-1\)

(b) \(-2\)

(c) Does not exist (since the answers to (a) and (b) do not match).

(d) 2

(e) 0

(f) Does not exist (since the answers to (d) and (e) do not match).

(g) 1 (the filled red dot above \( x = 2 \) is at \((2, 1)\), i.e. where \( y = 1 \).)

(h) 3 (everything is nice and smooth around \((4, 3)\)), so all limit exists, and they are all equal to the value of the function at \( t = 4 \), which is 3.
Problem Type 2.2.2: Determine the infinite limit

\[ \lim_{{x \to a^+}} \frac{\text{Top}(x)}{\text{Bottom}(x)}, \]

where \( \text{Bottom}(x) \) vanishes at \( x = a \) and \( \text{Top}(x) \) does not.

Example Problem 2.2.2: Determine the infinite limit

\[ \lim_{{x \to -2^+}} \frac{x - 1}{x^2(x + 2)}. \]

Steps

1. First make sure that indeed the bottom vanishes when you plug in \( x = a \), and the top does not. If they both vanish, you may need to use L’Hôpital’s rule (coming up later) first.

Example

1. \( \text{Top}(x) = x - 1, \)
   \[ \text{Bottom}(x) = x^2(x + 2). \]
   \[ \text{Top}(-2) = -3, \text{Bottom}(-2) = 0, \]
   so indeed the limit is going to be \( \infty \) or \( -\infty \).

2. Since we are looking for the limit from the right, plug in the expression a value very close to \( a \), but to its right, for example, \( a + .0001 \). You will either get something very negative or very positive. If it is very negative, then the answer is \( -\infty \), if it very positive, it is \( \infty \).

Example

2. \( \text{Top}(-1.9999) = -2.9999, \)
   \[ \text{Bottom}(-1.9999) = (-1.9999)^2(.0001), \]
   so the value of the expression at \( x = -1.9999 \) is
   \[ (-2.9999)/((-1.9999)^2(.0001)) \]
   which is VERY NEGATIVE (the exact value is not important).

Answer:

\[ \lim_{{x \to -2^+}} \frac{x - 1}{x^2(x + 2)} = -\infty \]