Some computations for $m$-dimensional partitions

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1. It was known to Euler that $p(n)$, the number of unrestricted partitions of $n$ into non-increasing integral parts, is generated by

$$
\sum_{n=0}^{\infty} p(n)x^n = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1} \ldots
$$

(1)

with the usual convention that $p(0) = 1$.

We may regard a partition of $n$ as an arrangement of nodes at integral points of the $x_1, x_2$ plane; thus

$$
10 = 5 + 3 + 1 + 1
$$

represented by

\[\begin{array}{c}
5 \\
3 \\
1 \\
1 \\
\end{array}\]

For $x_2 \rightarrow x_1$

This 'Ferrers-Sylvester graph' (cf. MacMahon (1), p. 3) represents a partition of $n$ into integers as a two-dimensional arrangement of nodes. We may form a natural generalization as follows.

By an 'unrestricted $m$-dimensional partition of $n$' we shall understand an arrangement of $n$ nodes at points of Euclidean $m$-space with non-negative integral coordinates, with the property that if a node $(a_1, a_2, \ldots, a_m)$ occurs then so also do all the nodes $(x_1, x_2, \ldots, x_m)$ with $0 \leq x_i \leq a_i$ ($i = 1, 2, \ldots, m$). We denote by $p_m(n)$ the number of distinct such partitions; trivially $p_1(n) = 1$ for all $n$. For $m \geq 2$ we compare $p_m(n)$ with $\pi_m(n)$ defined by

$$
\sum_{n=0}^{\infty} \pi_m(n)x^n = \prod_{r=1}^{m-3} (1-x^r)^{-\binom{r+m-3}{m-3}}
$$

(2)

where $\binom{r}{m}$ is the binomial coefficient with the usual conventions. Thus $p_2(n)$ is just the $p(n)$ of (1) above.
MacMahon (1) proved that \( p_3(n) = \pi_3(n) \), i.e.
\[
\sum_{n=0}^{\infty} p_3(n) x^n = (1-x)^{-1} (1-x^3)^{-2} (1-x^3)^{-3} \ldots,
\]
but both his proof and that of Chaundy (2) are difficult in comparison with the straightforward proof of (1).

Presumably MacMahon was aware that (2) did not enumerate partitions correctly for four or more dimensions (or, as he regarded it, for ‘solid partitions’ of numbers in three or more dimensions). Nanda (3, 4) assumes that \( p_4(n) = \pi_4(n) \) and writes down the form which MacMahon (1, p. 175) states ‘is shewn later not to be justified’. Thus in (4) Nanda tabulates \( \pi_4(n) \) and not \( p_4(n) \). Further work on the form of \( p_4(n) \) is found in (5).

It is natural to enquire what \( \pi_m(n) \) for \( m \geq 4 \) does enumerate in this context, and with this in mind we have computed a number of values of \( p_m(n) \) and \( \pi_m(n) \). The computation was carried out on a PDP 8 at Edinburgh University and on the Science Research Council’s Atlas I at Chilton; a description of the program and an Algo’ algorithm for \( p_m(n) \) by Bratley and McKay will appear elsewhere (6). The tiny required to compute \( p_m(n) \) from the combinatorial definition increases rapidly with \( m \) and \( n \), and in the absence of any clear conjecture from the first results we did not feel justified in using any more machine time. Writing
\[
E_m(n) = \pi_m(n) - p_m(n),
\]
we found the values of \( E_m(n) \) given in Table 2 at the end of this note.

2. If we now denote \( p_k^m(n) \) the number of unrestricted \( m \)-dimensional partitions of \( n \) whose nodes lie in some \( k \)-dimensional hyperplane but not in any \((k-1)\)-dimensional hyperplane, then we clearly have
\[
p_k^m(n) = 0 \quad \text{if} \quad k > m \quad \text{or} \quad k > n,
\]
and
\[
p_m(n) = \sum_{k=1}^{n-1} p_k^m(n) = \sum_{k=1}^{n-1} \binom{m}{k} p_k^m(n),
\]
\[
p_{n-1}^m(n) = 1.
\]
Thus, regarding \( p_m(n) \) as a function of \( m \) for fixed \( n \), we may write
\[
p_m(n) = \sum_{k=1}^{n-1} c_{kn} \binom{m}{k},
\]
where the \( c_{kn} \) are integers independent of \( m \), and \( c_{n-1,n} = 1 \). We also have from (2) that \( \pi_m(n) \) is a polynomial in \( m \) of degree \( (n-1) \) which takes integral values for \( m = 1, 2, \ldots, n-1 \), and so
\[
\pi_m(n) = \sum_{k=1}^{n-1} \gamma_{kn} \binom{m}{k},
\]
where the \( \gamma_{kn} \) are integers independent of \( m \), and it is easily seen that \( \gamma_{n-1,n} = 1 \). Hence
\[
E_m(n) = \pi_m(n) - p_m(n) = \sum_{k=1}^{n-1} c_{kn} \binom{m}{k},
\]
Some computations for m-dimensional partitions

where \( e_{n-1,n} = 0 \) from the above, and \( e_{kn} = 0 \) for \( 1 \leq k \leq 3 \) by Euler's and MacMahon's results. Thus finally

\[
E_m(n) = \sum_{k=1}^{n-2} e_{kn} \binom{m}{k},
\]

(7)

where the \( e_{kn} \) are integers independent of \( m \). A more tedious calculation shows that

\[
\gamma_{n-2,n} = 2^{n-3} + n - 3, \quad \text{while} \quad e_{n-2,n} = n - 2 + \binom{n-2}{2},
\]

so that

\[
e_{n-2,n} = 2^{n-3} - 1 - \binom{n-2}{2} = \sum_{k \geq 3} \binom{n-3}{k},
\]

(8)

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Using now the computed values in Table 1, we find

\[
E_m(n) = 0 \quad \text{if} \quad m \leq 3 \quad \text{or} \quad n \leq 5,
\]

\[
E_m(6) = \binom{m}{4},
\]

\[
E_m(7) = 3 \binom{m}{4} + 5 \binom{m}{5} = (m-1) \binom{m}{4},
\]

\[
E_m(8) = 8 \binom{m}{4} + 29 \binom{m}{5} + 16 \binom{m}{6},
\]

\[
E_m(9) = 19 \binom{m}{4} + 105 \binom{m}{5} + 145 \binom{m}{6} + 42 \binom{m}{7},
\]

\[
E_m(10) = 40 \binom{m}{4} + 321 \binom{m}{5} + 755 \binom{m}{6} + 545 \binom{m}{7} + 99 \binom{m}{8}.
\]

(9)
The results of (9), apart from MacMahon’s result for \( m = 3 \) and all \( n \), are of course somewhat trivial; the difficult problem is to determine what happens for fixed \( m \) and all \( n \). However, an immediate inquiry is whether \( E_m(n) > 0 \) for \( m \geq 4 \) and \( n \geq 6 \). For a fixed \( n \), this is certainly true for large enough \( m \) by (7) and (8). A stronger form of the question is:

Are the \( e_{km} \) in (7) always positive?

If so (and this seems to us likely), then it would appear that \( \pi_m(n) \) for \( m \geq 4 \) and \( n \geq 6 \) enumerates some additional objects which do not satisfy the original partition definition. A final question is whether, at any rate, \( \pi_m(n) \) gives the right order of magnitude for \( p_m(n) \), i.e.

Is \( E_m(n) = O(\pi_m(n)) \) valid for fixed \( m \) and \( n \to \infty \)?

The numerical evidence is insufficient to justify any conjecture.

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The non-zero values of \( E_m(n) \) are given below the values of \( \pi_m(n) \), which is easily computed. An asterisk denotes values deducible from other values using (7) and (8), which provided a check on the program.

REFERENCES