Multiplayer CHOMP

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Abstract: This article expands the famous problem CHOMP to multiplayer version. Since it is no longer a combinatorial game, its characteristics should be interesting and fruitful to discover. This article analyzes two specific kind of multiplayer Chomp game and yields satisfying results.

Key Words: Chomp, multiplayer, positions, scoring

Generic Multiplayer Chomp

We know that Chomp is a standard two-player combinatorial game. When it is extended to more players, it is natural to make those players to pick chocolate pieces in order. Different from win or lose, multiplayer game can value each player by scores. Players always seek for higher scores. For two-player game, winner gets 1 point and loser gets 0 points so valuing players by scores is reasonable.

Clockwise-Scoring Multiplayer Chomp

Assume that there are N players $(N \geq 3)$ numbered 1 through N play the Chomp game. The scale of the chocolate bar is $m \times n$ $(m, n \geq 1)$. The one who eats the last piece of chocolate scores 0 points, while his proceeding player scores 1 point, and the player after the proceeding player scores 2 points, and so on. The player before him scores $(N-1)$ points, which is the highest. All players want to maximize their scores. Who can get the most score in this multiplayer chomp game?

Start from the Simplest

We start from $N = 3$ and run some easy examples. **All the positions and graphs in this article are aligned to the top left.**

X	X
X	

Fig.1 The simplest example
(X means a piece of chocolate)

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As shown above, for this piece of L-shaped chocolate, player 1 has two options: pick all or pick one (note the symmetry!). Picking all yields 0 points while Picking one yields 1 point (player 2 can pick one too, so that he can get 2 points, see below). In order to maximize the score player 1 must choose the latter option letting player 2 get the most point.

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X    X          X             X
X               X
Initial Position -> Player 1 (1 pt)-> Player 2 (2 pts)-> Player 3 (0 pts)
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Fig.2 Playing the simplest example

To characterize all possible options for player 1, we can define an **option graph**, labeling scores player 1 could get on each chocolate piece.

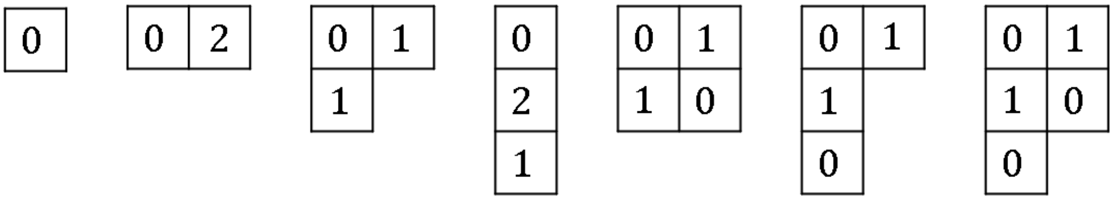

Fig.3 Option graph of several positions

Obviously, Player 1 will choose the chocolate piece with the highest score labeled on it.

Then We Go to $1 \times n$

We take a great leap to $1 \times n$ $(n \geq 1)$ with arbitrary N because it is easy to figure out.

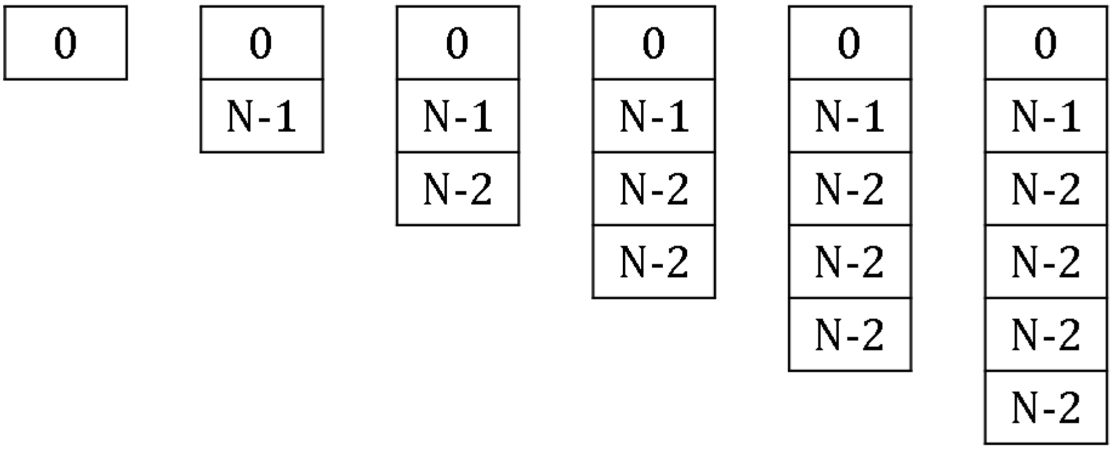

Fig.4 Option graph for $1 \times n$ $(n = 1,2,3,4,5,6)$

Here we show the reasons.

When $n = 1$, player 1 get 0 points and player N wins the highest points.
When $n = 2$, player 1 can either pick all and get 0 points or pick one and get $N - 1$ points. Obviously, player 1 will choose the latter and get the highest points.
When $n = 3$, player 1 can pick all (0 pts), pick two ($N - 1$ pts), or pick one ($N - 2$ pts). Obviously, player 1 will choose the second option and get the highest points.

Then we see a pattern: after picking up some chocolates, player 2 must pick the chocolate with the **highest** number labeled on **the position after player 1's move**. Then player 1's score (aka the number that should be labeled on the chocolate piece **of the original position**) should be exactly one less than player 2. But note that there is one exception: if player 2's score is 0, player 1 can get the highest score $N - 1$.
Therefore, by a simple deduction, we can easily figure out the answer for the **clockwise multiplayer chomp for $\mathbf{1} \times \boldsymbol{n}$ $(\boldsymbol{n} \geq \mathbf{1})$**:

player N wins the highest points when $n = 1$,
and player 1 wins the highest points for any $n \geq 2$.

March to Our Final Goal

With this crucial conclusion in hand, we can directly solve this multiplayer chomp once and for all. All we have left is a chocolate bar with scale $m \times n$ $(m, n \geq 2)$. However, to prove this situation, we should be greedy: we should get all possible positions involved. That is, the position need not to be rectangular. We call a position two-dimensional if it is NOT $1 \times n$ $(n \geq 1)$. For $1 \times n$ $(n \geq 2)$ position we call it one-dimensional. For 1×1 we call it zero-dimensional.
For a chocolate bar with scale $m \times n$ $(m, n \geq 2)$, the initial position is surely two-dimensional. We can not only prove that player 2 scores the most in rectangular chocolate bar, but also prove that player 2 scores the most in ANY two-dimensional positions!

We execute the deduction upon the total chocolate count T of the initial position.

(1) When $T = 3$, only forming L-shape can make the position two-dimensional. Player 1 has two options: pick away one chocolate remaining 1×2 position OR pick away all. The first option forces player 2 to pick away one, making player 3 achieve 0 points, so that player 2 can have $(N - 1)$ points. Player 1

can get $(N-2)$ points. The second option makes player 1 get 0 points. So, player 1 must choose the first option, making player 2 get the most points.

(2) When conclusion holds for every $T \leq k$, we consider $T = k + 1$.
Player 1 has three options:
Option 1: Take away all chocolates and get 0 points;
Option 2: Reduce the position to one-dimensional. Player 2 can reduce one-dimensional position to zero-dimensional so that player 3 get 0 points. Player 1 therefore get $(N-2)$ points;
Option 3: Remain the position on two-dimensional. Because after player 1's move, the chocolate count must be no more than k, so we can use the deduction assumption that player 3 gets the most points (because player 2 now executes the 'first' move). Player 1 therefore get $(N-3)$ points.
So, option 2 is the best choice for player 1, then player 2 can get the most score.

Thus, player 2 scores the most in ANY two-dimensional positions.

To visualize two-dimensional positions' option graph, we take 3×3 position as an example.

0	N-2	N-3
N-2	N-3	N-3
N-3	N-3	N-3

Fig.5 Example of two-dimensional positions' option graph

Counter-Clockwise-Scoring Chomp for 3 Players

Assume that there are 3 players numbered 1 through 3 play the Chomp game. The scale of the chocolate bar is $m \times n$ $(m, n \geq 1)$. The one who eats the last piece of chocolate scores 0 points, while his proceeding player scores 2 points, and the player before him scores 1 point. All players want to maximize their scores. Who can get the most score (aka 2) in this multiplayer chomp game?

Surprisingly, changing the direction of evaluating scores makes this problem much more difficult. Here we list some option graphs.

0

0
1

0	2
2	

0
1
2

0	2
2	0

0	0
2	
0	

0	0
2	0
0	

0	0	0
0		
0		

0	0
2	0
0	0

0	0	0
0	1	0
0	0	2

Fig.6 Some option graphs for 3-player counter-clockwise scoring chomp

Similar to clockwise version, we also have this in hand:
after picking up some chocolates, player 2 must pick the chocolate with the **highest** number labeled on **the position after player 1's move.** Then player 1's score (aka the number that should be labeled on the chocolate piece **of the original position**) should be exactly one **more** than player 2 **(not one less now).** But note that there is one exception: if player 2's score is 2, player 1 will get the score 0.

We can characterize a position as P-position or N-position if we are playing standard, two-player chomp. However, for 3-player chomp we can characterize a position as 0-, 1-, or 2-position, representing the highest score the NEXT player can get.

Option graph only contains 0 $\Leftrightarrow$ 0-position
Option graph only contains 0 and 1 $\Leftrightarrow$ 1-position
Option graph contains 2 $\Leftrightarrow$ 2-position

For any position, we can have a chocolate piece labeled 2 taken away to get a 1-position, and a chocolate piece labeled 1 taken away to get a 0-position, and a chocolate piece labeled 0 taken away to get a 2-position.

So, 0-position can only chomp to 2-position.

There is a special conclusion just for counter-clockwise scoring chomp.

If there exists a 2 in an option graph, all numbers labeled on its bottom-right (edges included) should be 0.

If player 1 takes away the piece on the bottom right of the piece labeled 2, player 2 can take the piece labeled 2 and get score 2. Then, player 1 will get a zero.

Walking the Tight Rope

Similar to standard two-player chomp, it is easy to analyze $1 \times n$ $(n \geq 1)$ and $2 \times n$ $(n \geq 2)$. Here we give the solution.

For $1 \times n$ $(n \geq 1)$, if $n = 1$, then player 2 wins; if $n = 2$, then player 3 wins; if $n = 3$, then player 1 wins. What if $n \geq 4$? We can guess the option graph of it should be like this:

0	1	2	0	0	0	0	•••	0

Fig.7 Option graph for $1 \times n$ $(n \geq 4)$

We can easily prove this through deduction.
Therefore, for $1 \times n$ $(n \geq 1)$,

if $n = 1$, then player 2 wins the highest points;
if $n = 2$, then player 3 wins the highest points;
if $n \geq 3$, then player 1 wins the highest points.

For $2 \times n$ $(n \geq 2)$, player 1 always wins because player 1 can directly chomp to 1×2 so that player 3 must take the last piece.

What about $3 \times n$?

0	1	2

0	2	0
0	0	0

0	0	0
0	1	0
0	0	2

0	0	0
0	2	0
0	0	0
0	0	0

Fig.8 Option graph for $3 \times n$ $(n = 1,2,3,4)$

It seems that player 1 always wins too. For odd n it is possible to get 0, 1, or 2 points for player 1; but for even n it is possible to get 0 or 2 points, but not 1 point. The secret behind remains unrevealed.

The Rigid 'L'

We are busy finding the winner for $3 \times n$ positions, but why don't we stop to find other positions we have already covered?

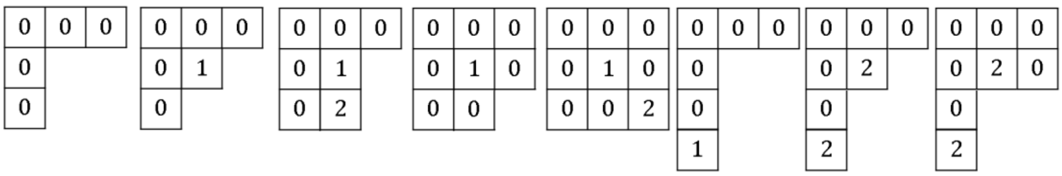

Fig.9 Discover the rigid 'L'

Chocolate pieces with coordinates (1,1), (1,2), (1,3), (2,1), (3,1) are always labeled 0. In fact, as long as this rigid 'L' exists, these five chocolate pieces must be labeled 0. The reason is as follows:

Operating (1,1) will take away all chocolate so player 1 scores 0;
Operating (1,2) or (2,1) will make the position $1 \times n$ $(n \geq 3)$, player 2 definitely can find a way to win 2 points, making player 1 scores 0;
Operating (1,3) or (3,1) will make the position $2 \times n$ $(n \geq 3)$, player 2 definitely can find a way to win 2 points, making player 1 scores 0;

So, the conclusion holds.

Cracking All 'L's

It is reasonable to figure out the option graphs of all L-shaped positions because of the presence of the rigid 'L'. It is shocking that we can use deduction to prove easily that option graphs of all L-shaped positions all follow a specific manner (for supporting information, see Appendix 1):

For an L-shaped position with leg length a, b, while $a \leq b$ (Typically, an 1×1 position have two legs, 1 and 1):

If $a = b$ and a is odd, all numbers in the option graph are 0;
If $a = b$ and a is even, all numbers in the option graph are 0, except that the 'tips' of the two legs are labeled 2;
If $a = b - 1$ and a is odd, all numbers in the option graph are 0, except that the 'tip' of the long leg is labeled 1;
If $a \leq b - 1$ and a is even, all numbers in the option graph are 0, except that the ath block of the long leg, counting from root to tip, is labeled 2;
If $a \leq b - 2$ and a is odd, all numbers in the option graph are 0, except that the $(a + 1)$th block of the long leg is labeled 1, and the $(a + 2)$th block of the long leg is labeled 2;

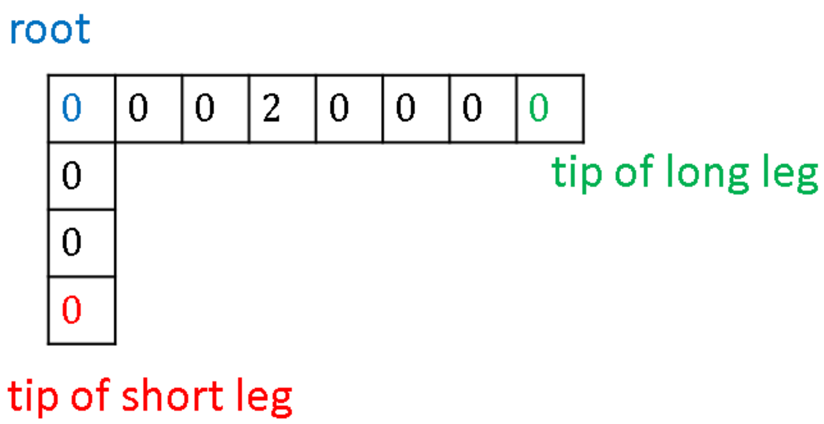

Fig.10 Legend of an L-shaped position

Note this rule: **If $\mathbf{a} = \mathbf{b} - \mathbf{1}$ and $\mathbf{a}$ is odd, all numbers in the option graph are 0, except that the 'tip' of the long leg is labeled 1.**
For a $(2k+1) \times (2k+2)$ scale chocolate (while k is a positive integer), player 1 can take away $(2k) \times (2k+1)$ and get an L-shape position with two legs $(2k+1)$ and $(2k+2)$. According to the previously mentioned rule, player 2 can get at most 1 point. Then, player 1 can get 2 points.

For a $(2\boldsymbol{k}+1) \times (2\boldsymbol{k}+2)$ scale chocolate (while $\boldsymbol{k}$ is a positive integer), player 1 wins the highest points.

Discovering the Stacking Hypothesis

What L-shape positions can tell us is that we can **stack chocolate pieces one by one** from a simple position and achieve the option graph of more complex positions. See Appendix 2 for examples.

Here we see a strange pattern: as the stacking grows sufficiently high, numbers in the same block seems to 'stabilize' and we are keep stacking 0. There tends to be only one '2' present and possibly one '1' – all other blocks are labeled 0.

We discover the **Stacking Hypothesis**:

For position $(A, a_2, a_3, a_4, \dots, a_m)$, as A grows from a_2 to infinity, there must exist a smallest $A = A_1$ such that for all $A \geq A_1$ all the previously-exist blocks remain the same labeled value and newly-added blocks will all be labeled 0.

See Appendix 3 for illustrations of this hypothesis. This stacking hypothesis is the extension of the L-shaped positions.

It is clear that the counter-clockwise chomp is much harder than the clockwise one. There are many patterns yet to be discovered and the final conclusion still remains unknown.

References

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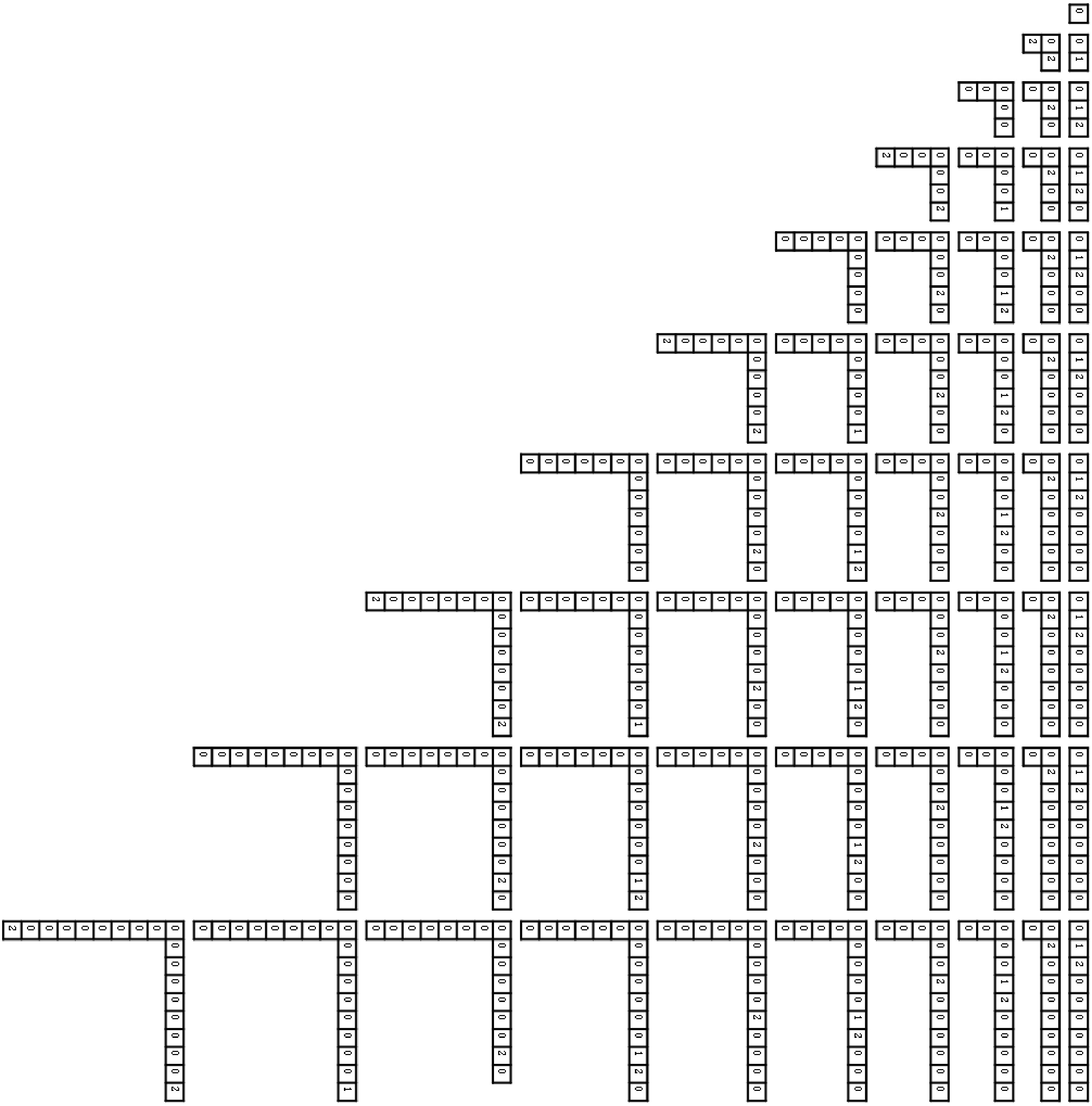
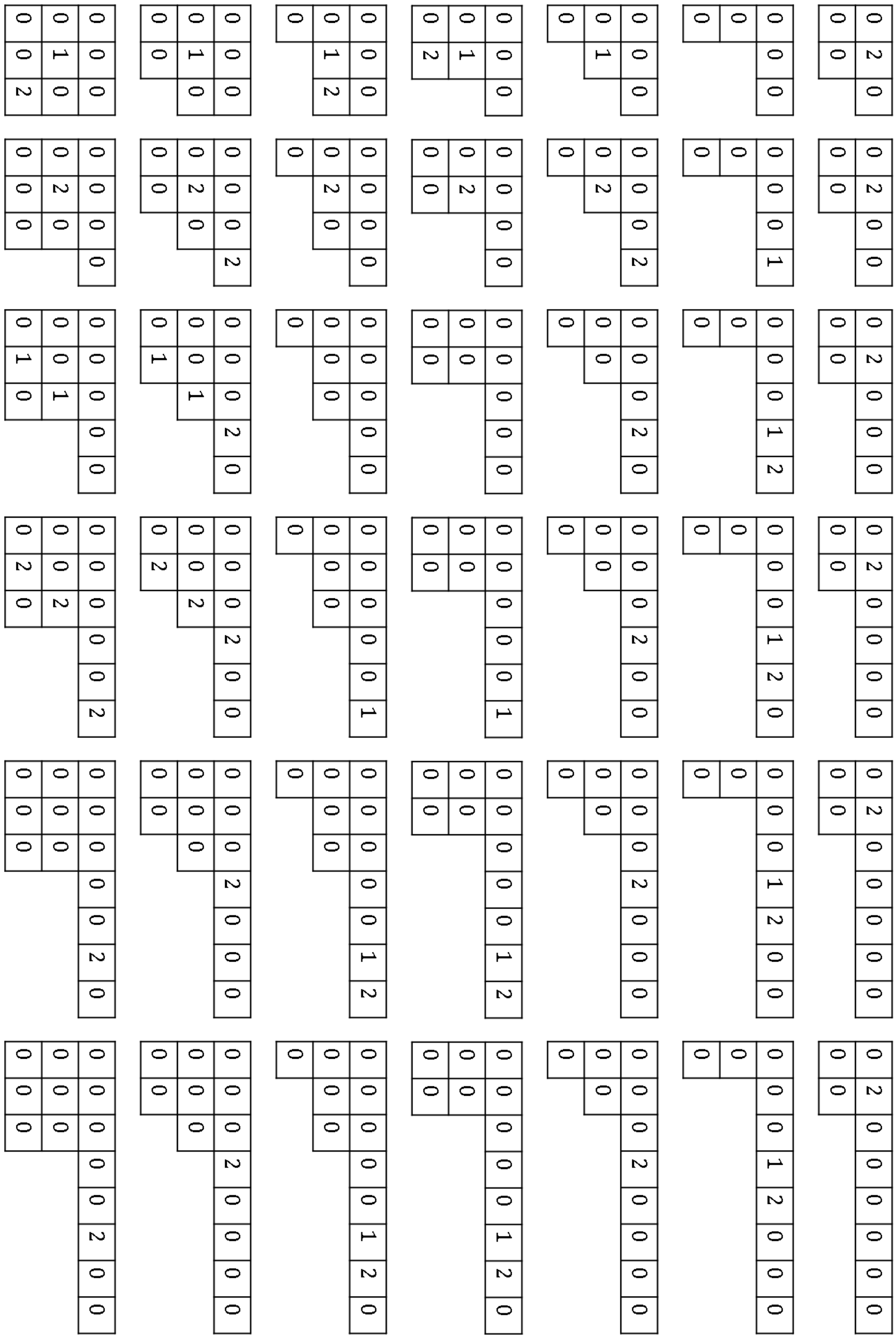
Appendix 1 Option graphs for some L-shaped positions



Appendix 2 Option graphs for some simple stacking



Appendix 3 Stacking Hypothesis

Positions	A_0	A_1	**Position list type***
(A)	1	3	012
$(A, 1)$		3	02
$(A, 2)$		3	02
$(A, 1,1)$	3	5	012
$(A, 2,1)$		5	02
$(A, 2,2)$	5	7	012
$(A, 3,1)$	5	7	012
$(A, 3,2)$		7	02
$(A, 3,3)$		7	02
$(A, 1,1,1)$		5	02
$(A, 1,1,1,1)$	5	7	012

A_0 satisfies that the resulting position is a 0-position.
*: 012 list type means that positions with sufficiently big A have exactly a 1 and a 2; 02 list type means that positions with sufficiently big A have exactly a 2.