A DIRECT PROOF OF AN INTEGRAL IDENTITY

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Recently, the integral identity

$$\int_0^1 \frac{x^n(1-x)^n}{\big((x+a)(x+b)\big)^{n+1}}\,dx = \int_0^1 \frac{x^n(1-x)^n}{\big((a-b)x+(a+1)b\big)^{n+1}}\,dx$$

was posted on arXiv[2]. We give a direct proof using a rational change of variables. Indeed, with

$$x = \frac{b(1-u)}{b+u}$$

we find that

$$dx = -\frac{b(1+b)}{(b+u)^2}\,du.$$

Moreover, 0 is mapped to 1 and 1 to 0. Thus, a direct calculation yields

$$\begin{aligned}\int_0^1 \frac{x^n(1-x)^n}{\big((x+a)(x+b)\big)^{n+1}}\,dx &= \int_0^1 \frac{\left(\frac{b(1-u)}{b+u}\right)^n\left(1-\frac{b(1-u)}{b+u}\right)^n}{\left[\left(\frac{b(1-u)}{b+u}+a\right)\left(\frac{b(1-u)}{b+u}+b\right)\right]^{n+1}}\frac{b(1+b)}{(b+u)^2}\,du\\ &= \int_0^1 \frac{(1-u)^n u^n b^{n+1}(1+b)^{n+1}}{\big[\big(b(1-u)+a(b+u)\big)\big(b(1-u)+b(b+u)\big)\big]^{n+1}}\,du\\ &= \int_0^1 \frac{u^n(1-u)^n}{\big((a-b)u+(a+1)b\big)^{n+1}}\,du.\end{aligned}$$

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2 See `https://arxiv.org/abs/1911.01423`