Rademacher on $J(\tau)$, Poincaré Series of Nonpositive Weights and the Eichler Cohomology

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Dedicated to Paul T. Bateman

This article is the twenty-second in the series of Special Articles published in the *Notices*. Marvin I. Knopp received the Ph.D. degree in 1958 from the University of Illinois (Urbana). His dissertation, written under the direction of Paul T. Bateman, dealt with modular forms of nonpositive weights. He has continued to study modular forms, focusing upon their Fourier coefficients, the Eichler cohomology theory and the theory of modular integrals and their Mellin transforms. He has held professorships at the University of Wisconsin, the University of Illinois at Chicago Circle, Bryn Mawr College and has been a Professor of Mathematics at Temple University since 1976.

I. Introduction.

1. Rademacher's perspective upon $J(\tau)$.
In 1939 Hans Rademacher presented a new construction of the modular invariant $J(\tau)$ in a seminal—yet little known—paper. This note examines Rademacher's construction from the perspective of a half-century of further advances in the theory of modular and automorphic forms. Especially important is the connection with the Eichler cohomology theory [3], developed some twenty years later.

The modular group $\Gamma(1)$, is the group of linear fractional transformations $V\tau = (a\tau + b)/(c\tau + d)$, τ complex, with $a, b, c, d \in Z$ (Z is the set of rational integers) and $ad - bc = 1$. The *absolute modular invariant* $J(\tau)$ is defined, for τ in the upper half-plane $\mathcal{H}$, by

$$(1) \qquad J(\tau) = 20G_4(\tau)^3/(20G_4(\tau)^3 - 49G_6(\tau)^2),$$

with

$$(2) \qquad G_k(\tau) = \sum_{m,n \in Z}{}' (m\tau + n)^{-k}, k \in Z, k \geq 3,$$

the *Eisenstein series of weight k*. (The notation $\sum'$ indicates omission of the term for $m = n = 0$.) The well-known behavior of $G_k(\tau)$ under $\Gamma(1)$ implies directly that $J(\tau)$ is invariant with respect to $\Gamma(1)$: $J(V\tau) = J(\tau)$, for all τ in $\mathcal{H}$ and $V \in \Gamma(1)$. (See [1, Chapter 1], [32, Chapter 3] and §I.2). Furthermore, among modular invariants $J(\tau)$ has the distinction that it generates the whole field of modular functions over the complex field C [17, Theorem 1E, p. 345]. It has the Fourier series expansion

$$(3) \qquad 12^3J(\tau) = e^{-2\pi i\tau} + 744 + \sum_{n=1}^{\infty} c_n e^{2\pi i n\tau}, \tau \in \mathcal{H},$$

where the c_n are rational integers [33, p. 56].

In 1938 Rademacher, unaware that Petersson [25] had already done so, published an exact formula for c_n [30]. (For specifics see (12).) Starting with that formula, in [31] he adopts an entirely fresh viewpoint concerning $J(\tau)$, taking it to be *defined* (anew) by (3) and the exact formula (12). He poses the problem: to show *from this new definition* that $J(\tau)$ is a modular invariant. Since the two transformations $S\tau = \tau + 1$, $T\tau = -1/\tau$ generate $\Gamma(1)$ [12, p. 7] and since $J(\tau + 1) = J(\tau)$ follows directly from (3), this problem reduces (an odd word, considering the difficulty involved) to that of deriving, from (3) and (12) alone, the equation $J(-1/\tau) = J(\tau)$, now far from obvious. Rademacher solves this problem by carrying out a profound transformation of the function defined by (3) and (12), representing $J(\tau)$ as what we may now term a "modified Poincaré series."

2. Poincaré's construction of automorphic functions.
Poincaré series appear for the first time in Poincaré's celebrated 1882 memoir on Fuchsian functions [28]. They provide a perspective indespensible for understanding Rademacher's construction and the later work [8, 9, 10, 11, 21, 35] based directly upon it.

Poincaré deals with groups of linear fractional transformations acting on the unit disc, in particular, with the construction of their invariant functions. For consistency with our introductory remarks we consider instead groups Γ of linear fractional transformations acting on the upper half-plane $\mathcal{H}$; that is, we assume that Γ is a discrete group of mappings $V\tau = (a\tau + b)/(c\tau + d)$, with a, b, c, d real and $ad - bc > 0$. If Γ is finite, it is an easy matter to construct a meromorphic function

$F(\tau)$, *automorphic* (that is, invariant) *with respect to* Γ, by forming the finite sum

$$F(\tau) = \sum_{V \in \Gamma} f(V\tau), \tag{4}$$

with f chosen meromorphic in $\mathcal{H}$. That F does not reduce to a constant can be guaranteed by a suitable choice of f.

When Γ is infinite, in contrast, this simple construction does not suffice, because the series in (4) may fail to converge. To overcome this difficulty, Poincaré introduced the series which bear his name:

$$F_k(\tau; f) = \sum_{V \in \Gamma} \frac{f(V\tau)}{(c\tau + d)^k}, V\tau = (a\tau + b)/(c\tau + d); \tag{5}$$

in (5) f is a rational function and k a positive integer chosen large enough to guarantee absolute-uniform convergence of the series (5) in compact subsets of $\mathcal{H}$. (The existence of such k follows from the discreteness of Γ.) The function $F_k(\tau; f)$ so formed, while meromorphic in $\mathcal{H}$, fails to have the desired simple automorphic property

$$F(M\tau) = F(\tau), \text{ all } M \in \Gamma, \tag{6}$$

characteristic of functions F defined by (4). However, the absolute convergence of (5) implies readily that for all $M = \left(\begin{smallmatrix} * & * \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$,

$$F_k(M\tau; f) = (\gamma\tau + \delta)^k F_k(\tau; f), \tag{7}$$

since $(\gamma\tau + \delta)^k (cM\tau + d)^k = (c'\tau + d')^k$, where $c'\tau + d'$ is the lower row of VM. A function satisfying the transformation formulae (7) and certain regularity conditions is called an *automorphic form of weight k on* Γ. (We note that the Eisenstein series $G_k(\tau)$, defined by (2), is virtually the same as the Poincaré series (5), with $f \equiv 1$ and $\Gamma = \Gamma(1)$. In any event, $G_k(\tau)$ is a modular form—that is an automorphic form on $\Gamma(1)$—of weight k, since $G_k(\tau)$ satisfies (7) for $M \in \Gamma(1)$.) A function satisfying (6) is called an *automorphic function on* Γ.

To reach his original goal, construction of nontrivial meromorphic functions possessing the (absolute) invariance property (6) with respect to Γ, Poincaré forms the quotient $F_k(\tau; f_1)/F_k(\tau; f_2)$, with rational f_1 and f_2 chosen to have distinct singularities in $\mathcal{H}$. The latter condition ensures that this quotient does not reduce to a constant.

This work of Poincaré on the problem of constructing automorphic (he called them "Fuchsian") functions provides the context for the well-known story of his sudden revelation while stepping on a bus to go on an excursion, and in the midst of an unrelated conversation. The unexpected insight was the relationship of his (Poincaré) series (5) to the rigid motions of hyperbolic geometry. This revelation took place immediately following a two week period during which he thought intensely about the question, but with inconclusive partial results. Poincaré ultimately obtained a complete solution, but only after several further similar occurrences, equally unexpected and sudden. Of particular interest is Poincaré's immediate recognition, in each instance, that the new idea would be fruitful, before working through any of the details. For Poincaré's own account see [29, pp. 52–55]. There is a discussion of this episode as well in [5, pp. 12–15].

3. Elliptic functions and Eisenstein series.

A problem different in technical detail, but virtually identical in spirit, is one resolved before Poincaré's birth: that of constructing elliptic functions, that is to say, functions meromorphic in the complex plane C and automorphic with respect to a group of translations in two independent directions. Suppose ω_1 and ω_2 are complex numbers with $\mathrm{Im}(\omega_1/\omega_2) \neq 0$. Then the "lattice" $L = \{m\omega_1 + n\omega_2 | m, n \in Z\}$ is discrete in C and when $k \geq 3$ the "Mittag-Leffler sum",

$$E_k(z) = E_k(z; \omega_1, \omega_2) = \sum_{m,n \in Z} \frac{1}{(z + m\omega_1 + n\omega_2)^k}, \tag{8}$$

represents a function (in fact, the simplest one) with a pole of order k at each of the lattice points. When $k \geq 3$ the series (8) obviously converges to a function meromorphic in C, but much more is true. The Mittag-Leffler construction, designed only to produce meromorphic functions with poles at a prescribed discrete set of points (with prescribed principal parts, as well), actually yields elliptic functions invariant with respect to the group $G = G(\omega_1, \omega_2) = \{z \to z + \omega | \omega \in L\}$. This obtains since (8) can be rewritten, in analogy with (4) and (5), as

$$E_k(z) = \sum_{V \in G} f_k(Vz), f_k(z) = z^{-k}. \tag{9}$$

Here, once again, the Eisenstein series $G_k(\tau)$ come to mind, for defining

$$E_k^*(z; \omega_1, \omega_2) = E_k(z; \omega_1, \omega_2) - \frac{1}{z^k},$$

we have $E_k^*(0; \tau; 1) = G_k(\tau)$.

For $k = 2$ the nice convergence (absolute-uniform on compact subsets of C which do not contain any of the lattice points $m\omega_1 + n\omega_2$) of (9) fails, but—as is well understood—this difficulty is inessential, easily overcome by subtraction of "convergence terms" from the summands:

$$E_2(z) = \frac{1}{z^2} + \sum_{m,n \in Z}{}' \left\{ \frac{1}{(z + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right\}. \tag{10}$$

(Here, $\sum'$ has the same meaning as in (2).) "Nice" convergence of (10) follows readily, and $E_2(z)$ clearly has a pole of order 2 at each lattice point. However, in re-establishing convergence we apparently have sacrificed invariance with respect to $G(\omega_1, \omega_2)$. It appears, in fact, that invariance has given way to the characteristic functional equation of an elliptic integral:

$$(11) \qquad E_2(Vz) = E_2(z) + C_V, V \in G(\omega_1, \omega_2),$$

where C_V is a constant dependent upon V. As it turns out, the sacrifice is only apparent, since $E_2(z)$ is an even function ($E_2(-z) = E_2(z)$) and this fact implies directly that $C_V = 0$ for all $V \in G$. Thus, $E_2(z)$ is a genuine elliptic function, not an elliptic integral. Of course, $E_2(z)$ is the well-known Weierstrass function $\wp(z)$ [7,§9].

4. Remarks on automorphic forms.
The modular invariant $J(\tau)$ was first studied by Dedekind and, independently, by Klein in 1877, about fifty years after the development of elliptic functions. Shortly thereafter, Picard used $J(\tau)$ and the monodromy theorem to prove his famous "little" theorem: an entire function omitting more than one complex value from its range is constant. Nowadays Picard's little theorem is most often proved using instead of $J(\tau)$ the related function $\lambda(\tau)$, invariant with respect to the principal congruence subgroup of level 2, a normal subgroup of index 6 in $\Gamma(1)$. ($\lambda(\tau)$ has a simple expression in terms of the Weierstrass elliptic functions $\wp(z)$ and $\wp'(z)$.) About the same time H. A. Schwarz and Poincaré used the theory of automorphic functions in studying ordinary second-order linear differential equations.

This was the context for Poincaré's fundamental work [28], which effectively initiated a systematic theory of automorphic forms and automorphic functions with respect to *Fuchsian groups*, discrete groups of linear fractional transformations acting on a half-plane or disc. In the background of Poincaré's work (and dominant in the contemporaneous work of Klein) is the idea of a Riemann surface. In light of the later "uniformization theorem," proved completely in 1912 after thirty years of effort by a number of mathematicians, the theory of Riemann surfaces can be viewed as contained in the theory of Fuchsian groups. The relationship between the two theories is close at hand: given a Fuchsian group Γ acting on a disc or half-plane D, one can introduce a natural topology on the set of orbits $S = \Gamma\backslash D$ in such a way that the structure is analytic, the topological space S a Riemann surface. Then the meromorphic functions on S correspond to the automorphic functions with respect to Γ, the (first-order) differentials on S to the automorphic forms of weight 2 with respect to Γ.

The problem of uniformization deals with the converse: given an arbitrary Riemann surface is there a pair (Γ, D) as above, such that S is conformally equivalent to $\Gamma\backslash D$? The affirmative answer given by the uniformization theorem means that the theory of Fuchsian groups is coextensive with the theory of Riemann surfaces. However, there does not seem to be a simple, natural interpretation of automorphic forms of arbitrary real weights on a Fuchsian group in terms of the corresponding Riemann surface.

For further details about the history of automorphic forms and their role in contemporary mathematics, I refer the reader to the Historical Development chapter (chapter 1) of [17] and to the references supplied there.

II. Rademacher's work on $J(\tau)$.

1. $J(\tau)$ as a parabolic Poincaré series.
The exact formula of Petersson and Rademacher for the coefficients c_n in the expansion (3) of $J(\tau)$ is

$$(12) \qquad c_n = \frac{2\pi}{\sqrt{n}} \sum_{\ell=1}^{\infty} \ell^{-1} A_\ell(n) I_1\left(\frac{4\pi\sqrt{n}}{\ell}\right), n \geq 1.$$

Here, $A_\ell(n)$ is a Kloosterman sum defined by

(13)
$$A_\ell(n) = \sum_{\substack{h \pmod{\ell} \\ (h,\ell)=1}} \exp\left[\frac{-2\pi i}{\ell}(nh + h')\right], hh' \equiv -1 \pmod{\ell},$$

while I_1 is the modified Bessel function of the first kind, given by the power series

$$(14) \qquad I_1(x) = \sum_{j=0}^{\infty} \frac{(x/2)^{2j+1}}{j!(j+1)!} .$$

Rademacher obtains this formula in [30] by a refinement of his own variant [32] of the Hardy-Ramanujan "circle method" (in contrast to Petersson's entirely different approach involving modular forms of weight 2). In [31] he proves the following result, which brings to light a completely new way of viewing the fundamental modular invariant $J(\tau)$.

Theorem 1. [31, (4.1)]. *$J(\tau)$ has the representation*

(15)
$$\begin{aligned} 12^3 J(\tau) =& e^{-2\pi i\tau} + e^{2\pi i/\tau} + 731 \\ &+ \lim_{K\to\infty} \sum_{\ell=1}^{K} \sum_{\substack{1\leq|m|\leq K \\ (\ell,m)=1}} \left\{\exp\left(-2\pi i \frac{m'\tau + \ell'}{\ell\tau - m}\right)\right. \\ &\left. - \exp(-2\pi i m'/\ell)\right\}, \end{aligned}$$

where m' is any integral solution of the congruence $mm' \equiv -1 \pmod{\ell}$ and ℓ' is the rational integer defined by $-\ell' = (mm' + 1)/\ell$.

In the historical introduction to his influential work, *Discontinuous groups and automorphic functions*, J. Lehner has expressed the opinion that this formula for $J(\tau)$ "is as striking and elegant as the classic identities of Euler and Jacobi". [17, p. 41] Lehner further refers to the formula (15) as "an expansion of J into partial fractions," and he compares it with the definition (10) of the Weierstrass function $\wp$, noting in particular, "the subtracted 'convergence summand' in each case" [17, pp. 40–41]. To appreciate more fully the insight which informs this remark, we rewrite (15) to resemble closely a modified form of the Poincaré series (5), namely, the "parabolic Poincaré series" introduced by Petersson [24]:

$$G_k(\tau;\nu) = \sum\nolimits^{*} \frac{e^{2\pi i \nu V\tau/\lambda}}{(c\tau+d)^k}. \tag{16}$$

Here, ν is an arbitrary integer, k is an integer, $\lambda > 0$ is the minimal width of a translation in Γ (e.g., $\lambda = 1$ for $\Gamma = \Gamma(1)$) and the notation $\sum^{*}$ indicates that—in contrast to the summation over all $V \in \Gamma$ as in (5)—in (16) the sum is confined to $V \in \Gamma$ with distinct lower row c, d. This restriction arises naturally as a necessary condition for convergence since the rational function f of (5) has given way in (16) to the exponential function $e^{2\pi i\nu\tau/\lambda}$. For, the periodicity of $e^{2\pi i\nu\tau/\lambda}$ implies that the numerator $e^{2\pi i\nu V\tau/\lambda}$ in (16) is independent of the upper row of V, and from this it follows directly that, regardless of the size of k, the full sum on all $V \in \Gamma$ cannot converge, since each summand would then occur infinitely often. On the other hand, the assumption $k > 2$ assures absolute uniform convergence of the series in (16) on compact subsets of $\mathcal{H}$ [17, pp. 276–277].

As with the Poincaré series (5), Petersson's modified Poincaré series (16) are automorphic forms of weight k on Γ as long as k is an integer greater than two. The proof is the same in both cases. The parabolic Poincaré series have two clear advantages over the Poincaré series (5): (i) the analytic behavior of $G_k(\tau;\nu)$ can be controlled completely at the parabolic cusps of Γ; (ii) the $G_k(\tau;\nu)$ behave well with respect to the Petersson inner product, well enough, indeed, to make possible a direct inference that they form a basis for all automorphic forms of weight k on Γ which are holomorphic in $\mathcal{H}$ and at the finite cusps [17, pp. 284–289].

To compare the expression (15) with the parabolic Poincaré series (16) we begin by recalling that the full modular group $\Gamma(1)$ is the group of invariance for $J(\tau)$. Furthermore, $c\tau + d$ occurs as the lower row of a transformation in $\Gamma(1)$ precisely when c and d are relatively prime integers. With a simple change of notation the double sum in (15) becomes

$$\sum_{1\le c\le K} \sum_{\substack{1\le |d|\le K\\ (c,d)=1}} \left\{ \exp\left(-2\pi i \frac{a\tau+b}{c\tau+d}\right) - \exp(-2\pi i a/c) \right\}, \tag{17}$$

where a and b are so chosen that $V_{c,d} = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma(1)$. We next take advantage of the fact that the summands in (17) are unchanged under replacement of the pair (c,d) by the pair $(-c,-d)$, to rewrite the sum (17) as

$$\frac{1}{2} \sum_{1\le|c|\le K} \sum_{\substack{1\le |d|\le K\\ (c,d)=1}} \{\exp(-2\pi i V_{c,d}\tau) - \exp(-2\pi i a/c)\}.$$

But, the condition $(c,d) = 1$ implies that $c = 0$ occurs only with $d = \pm 1$, and $d = 0$ only with $c = \pm 1$. In the former case we can choose the matrices to be $\pm\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)$, and in the latter, $\pm\left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)$. Thus, with the definition

$$s(c,d) = \begin{cases} e^{-2\pi i a/c} & c \ne 0\\ 0, & c = 0, \end{cases}$$

the expression (15) becomes

$$12^3 J(\tau) = 732 + \lim_{K\to\infty} \frac{1}{2} \sum_{|c|\le K} \sum_{\substack{|d|\le K\\ (c,d)=1}} \{\exp(-2\pi i V_{c,d}\tau) - s(c,d)\}. \tag{18}$$

Comparison of (18) with (16) now clearly justifies our viewing the right-hand side of (18) as a parabolic Poincaré series of weight $k = 0$ on the group $\Gamma(1)$, for $\nu = -1$, but with the order of summation prescribed explicitly (lattice points in expanding squares) and modified by the subtracted convergence summands $s(c,d)$. We stress that without this prescription and modification, there can be no hope of convergence for a Poincaré series of weight $k = 0$. (For $\Gamma(1)$ absolute convergence occurs only if $k > 2$.) Even with them, convergence remains far from obvious. The proof comes naturally out of Rademacher's development.

2. Invariance of $J(\tau)$.

As we remarked earlier in reference to the definition (10) of $\wp(z)$ $(= E_2(z))$, modification by the convergence summands $s(c,d)$ seems to disturb the invariance under $\Gamma(1)$ of the right-hand side of (18), leading to the introduction of additive periods in the transformation formulae (as in (11)). At first glance, then, Rademacher's formula (18) appears to imply only that $J(\tau)$ is an abelian integral with respect to $\Gamma(1)$, but not necessarily a modular invariant (i.e. that the periods vanish). But, as in the case of $\wp(z)$, invariance does follow quite directly from the structure of the formula (18).

To see this, recall that invariance under all of $\Gamma(1)$ will result from the single transformation formula $J(-1/\tau) = J(\tau)$. We note from (18) that to prove this it will suffice to demonstrate the invariance under $\tau \to -1/\tau$ of the finite sum

$$\sum\nolimits_K(\tau) = \sum_{|c|\le K} \sum_{\substack{|d|\le K\\ (c,d)=1}} \exp(-2\pi i V_{c,d}\tau).$$

However,

$$\sum\nolimits_K(-1/\tau) = \sum_{|c|\le K}\sum_{\substack{|d|\le K\\(c,d)=1}} \exp\{-2\pi i V_{c,d}(-1/\tau)\}$$
$$= \sum_{|c|\le K}\sum_{\substack{|d|\le K\\(c,d)=1}} \exp(-2\pi i V_{d,-c}\tau),$$

since $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix} b & -a\\ d & -c\end{pmatrix} = V_{d,-c}$.

Now we infer $\sum_K(-1/\tau) = \sum_K(\tau)$ by matching the pair (c,d) with the pair $(d,-c)$.

3. Sketch of the proof of Theorem 1.

We present a very brief account of Rademacher's derivation of the expression (15) from the definition of $J(\tau)$ given by (3) and (12). At the heart of the method is a difficult technical lemma justifying rearrangement of a certain conditionally convergent double series.

Rademacher begins by inserting the expressions (12) and (13) into (3), and then inverting the order of summation in the double sum so obtained. The validity of this step relies crucially upon the estimate of Weil [36],

$$A_\ell(n) = \mathcal{O}(\ell^{1/2+\epsilon}), \epsilon > 0,$$

uniformly in n. (Actually, any nontrivial estimate $A_\ell(n) = \mathcal{O}(\ell^{1-\delta}), \delta > 0$, would suffice for the purpose.) The interchange of summations implies that $12^3 J(\tau)$, as defined by (3) and (12), equals

(19)
$$e^{-2\pi i\tau} + 744 + 2\pi\sum_{\ell=1}^{\infty}\frac{1}{\ell}\sum_{h(\mathrm{mod}\,\ell)} e^{-2\pi i h'/\ell} \times \sum_{n=1}^{\infty}\exp[2\pi i n(\tau - h/\ell)]\frac{1}{\sqrt{n}} I_1\left(\frac{4\pi\sqrt{n}}{\ell}\right).$$

Next, he replaces I_1 by the power series (14), performs another interchange of summations and—in the salient feature of his calculation—makes use of the Lipschitz sum formula [12, p. 65],

(20)
$$\sum_{n=1}^{\infty} n^p e^{2\pi i n\tau} = \begin{cases} -\frac{1}{2} + \frac{1}{2\pi}\lim_{N\to\infty}\sum_{q=-N}^{N}(-i\tau + iq)^{-1}, & p = 0\\ \frac{p!}{(2\pi)^{p+1}}\sum_{q=-\infty}^{\infty}(-i\tau + iq)^{-p-1}, & p\in Z^+.\end{cases}$$

This transforms (19) into

(21)
$$e^{-2\pi i\tau} + 732 + \sum_{\ell=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{|m|\le N\\(m,\ell)=1}} e^{-2\pi i m'/\ell}\sum_{p=1}^{\infty}\frac{1}{p!}\left\{\frac{2\pi i}{\ell(\ell\tau - m)}\right\}^p.$$

At this point Rademacher divides the multiple sum in (21) into the two parts

$$\sum_{\ell=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{|m|\le N\\(m,\ell)=1}} e^{-2\pi i m'/\ell}\left\{\frac{2\pi i}{\ell(\ell\tau - m)}\right\}$$
$$+\sum_{\ell=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{|m|\le N\\(m,\ell)=1}} e^{-2\pi i m'/\ell}\sum_{p=2}^{\infty}\frac{1}{p!}\left\{\frac{2\pi i}{\ell(\ell\tau - m)}\right\}^p,$$

a step justified on the grounds that the second is absolutely convergent as a triple sum and first is convergent by virtue of its appearance as the left-hand side in

Rademacher's Lemma. [31, p. 238, (2.1)]. *Suppose $\tau\in\mathcal{H}$. Then,*

(22)
$$\sum_{\ell=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{|m|\le N\\(m,\ell)=1}}\left\{\frac{\exp(-2\pi i m'/\ell)}{\ell(\ell\tau - m)}\right\} = \lim_{K\to\infty}\sum_{\ell=1}^{K}\sum_{\substack{|m|\le K\\(m,\ell)=1}}\{\},$$

with m' defined as in the statement of Theorem 1. (Convergence of the right-hand side of (22) implies directly the convergence of (18), the (modified) parabolic Poincaré series of weight $k = 0$.)

Applying the Lemma to the first sum and absolute convergence to the second, he obtains

(23)
$$12^3 J(\tau) = e^{-2\pi i\tau} + 732 + \lim_{K\to\infty}\sum_{\ell=1}^{K}\sum_{\substack{|m|\le K\\(m,\ell)=1}} e^{-2\pi i m'/\ell}\sum_{p=1}^{\infty}\frac{1}{p!}\left\{\frac{2\pi i}{\ell(\ell\tau - m)}\right\}^p$$
$$= e^{-2\pi i\tau} + 732 + \lim_{K\to\infty}\sum_{\ell=1}^{K}\sum_{\substack{|m|\le K\\(m,\ell)=1}} e^{-2\pi i m'/\ell}\left\{\exp\left(\frac{2\pi i}{\ell(\ell\tau - m)}\right) - 1\right\}.$$

Finally, (15) results from (23) upon separation of the single term for $m = 0$ (with $\ell = 1$), and application of the simple identity

$$-m'/\ell + \frac{1}{\ell(\ell\tau - m)} = \frac{-m'\tau - \ell'}{\ell\tau - m}$$

(m', ℓ' defined as in the statement of Theorem 1) and the invariance of the summand under the map $(\ell, m)\to(-\ell,-m)$.

III. Connection with Eichler cohomology.

1. Generalization to modular forms of nonpositive weight. A number of mathematicians have developed Rademacher's ideas further, extending them (i) to discrete groups of real linear fractional transformations other than $\Gamma(1)$; (ii) to automorphic forms of weights $k \le 0$. (See §I.2, following (7), for the definition.) Here we emphasize the generalization to forms of negative weight, as this leads directly to the Eichler cohomology theory.

We confine our attention to the case of automorphic forms F on $\Gamma(1)$ (that is, *modular forms*) of weight $k \le 0$, with k an even integer and with "multiplier system" identically one. This means F satisfies (7), with k even and ≤ 0, for all $M \in \Gamma(1)$. (For a definition and discussion of multiplier systems, see [12, pp. 12–13] or [17, pp. 267–268].) The definition of modular form requires, as well, that F be holomorphic in $\mathcal{H}$ and expressible there as an exponential series of the form

$$F(\tau) = \sum_{n=-\mu}^{\infty} a_n e^{2\pi i n\tau}. \tag{24}$$

(Note that periodicity of F follows from (7), with $M = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.)

Rademacher derived the exact formula (12) for the coefficients in the expansion of $J(\tau)$ by refining a method he and Zuckerman [32] had devised earlier to calculate the coefficients a_n in the exponential series (24) for an arbitrary modular form F of negative weight k ("positive dimension"—k in the terminology of [32]). The formula of Rademacher and Zuckerman for the a_n, in the special case when k is even (and the multiplier system is identically one), is

$$\begin{aligned} a_n = (-1)^{k/2}(2\pi) \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{\ell=1}^{\infty} \\ \ell^{-1} A_{\ell,\nu}(n)(\nu/n)^{(-k+1)/2} I_{-k+1}\left\{\frac{4\pi}{\ell}\sqrt{n\nu}\right\}, n \ge 1, \end{aligned} \tag{25}$$

where

$$A_{\ell,\nu}(n) = \sum_{\substack{h \pmod{\ell} \\ (h,\ell)=1}} \exp\left[\frac{-2\pi i}{\ell}(nh + \nu h')\right], hh' \equiv -1 \pmod{\ell}, \tag{26}$$

and

$$I_{-k+1}(x) = \sum_{j=0}^{\infty} \frac{(x/2)^{2j-k+1}}{j!(j-k+1)!}. \tag{27}$$

Remarks. 1. The expression (25) implies that a modular form of weight $k < 0$ which is bounded at $i\infty$ (that is, $a_{-1} = a_{-2} = \cdots = a_{-\mu} = 0$ in (24)) is identically zero. This can be proved for $k \le 0$ without invoking (25) ([12, pp. 24–30] or [17, pp. 166–176]).

2. When $k = 0$, $\mu = 1$ and $a_{-1} = 1$, then (25) reduces to the series (12) for the coefficient of $J(\tau)$.

3. Both Lehner [19, 20] and Petersson [23] have derived (25) in the broader setting in which a general H-group Γ [17, p. 266] replaces $\Gamma(1)$. In this more general case, the structure of the series for a_n remains unchanged in its essentials.

The Rademacher-Zuckerman formula (25) makes available in the wider context of modular forms with weights $k \le 0$ precisely the same viewpoint that Rademacher adopts in [31] toward $J(\tau)$. Specifically, one can *define* the function $F(\tau)$ by the series (24) and (25), and ask—as Rademacher did concerning $J(\tau)$—how (or, more tentatively, whether) it is possible to show from this definition that $F(\tau)$ does in fact satisfy (7), the characteristic functional equation for a modular form of weight k. This perspective, indeed, forms the basis for much of the work I undertook in the early 1960's [8, 9, 10, 11, 13, 14]. As it turns out, it is impossible to show that all functions $F(\tau)$ so defined are modular forms, not because of a defect in the method, but rather because they are not all modular forms. We shall comment upon this further in §III.2, below.

Rademacher's method yields:

Theorem 2. [11, p. 28, (3.07)]. *For $\tau \in \mathcal{H}$ define the function*

$$F_\nu(\tau) = e^{-2\pi i\nu\tau} + (-1)^{k/2}(2\pi)\sum_{n=1}^{\infty} a_n(\nu) e^{2\pi i n\tau},$$

with ν a positive integer and $a_n(\nu)$ the infinite sum on ℓ occurring in (25). Put $r = -k$, a positive even integer. Then, $F_\nu(\tau)$ is holomorphic in $\mathcal{H}$ and it there has the representation

$$\begin{aligned} F_\nu(\tau) = e^{-2\pi i\nu\tau} + \alpha_\nu + \tau^r \left\{\exp(2\pi i\nu/\tau) - \sum_{t=0}^{r} \frac{1}{t!}(2\pi i\nu/\tau)^t\right\} \\ + \lim_{K\to\infty} \sum_{\ell=1}^{K} \sum_{1\le|m|\le K} (\ell\tau - m)^r \left\{\exp\left(-2\pi i\nu\frac{m'\tau+\ell'}{\ell\tau - m}\right) \right. \\ \left. - \exp(-2\pi i m'\nu/\ell)\sum_{t=0}^{r}\frac{1}{t!}\left(\frac{2\pi i\nu}{\ell(\ell\tau-m)}\right)^t\right\}, \end{aligned} \tag{28}$$

where α_ν is a constant depending only upon ν and r. (See [11, p. 27], where α_ν is denoted c_ν.)

The proof of (28) requires that a generalization of the Rademacher Lemma in which

$\ell^{1+r}(\ell\tau - m)$ $(= \ell^{1-k}(\ell\tau - m))$ replaces the denominator $\ell(\ell\tau - m)$ of (19). The analogy of (28) with (15) needs no clarification; when $\nu = 1$ and $k = 0$ (so $r = 0$), (28) reduces to (15). Like the expression (15) for $J(\tau)$, (28) can be rewritten as a modified parabolic Poincaré series of weight $k < 0$ on $\Gamma(1)$, comparable to (18):

(29)
$$F_\nu(\tau) - \alpha_\nu = \lim_{K\to\infty} \sum_{|c|\le K} \sum_{\substack{|d|\le K \\ (c,d)=1}} \{(c\tau + d)^r \exp(-2\pi i \nu V_{c,d}\tau) - q(\tau; c, d)\},$$

where $q(\tau; c, d)$ is the polynomial of degree $r = -k$ given by

(30)
$$q(\tau; c, d) = \begin{cases} e^{-2\pi i a\nu/c} \sum_{l=0}^{r} \frac{1}{l!}\left(\frac{2\pi i \nu}{c}\right)^l (c\tau + d)^{r-l}, & c \neq 0, \\ 0, & c = 0. \end{cases}$$

The "convergence terms" $q(\tau; c, d)$ here replace the $s(c, d)$ appearing in (18); when $\nu = 1$ and $k = 0$, $q(\tau; c, d)$ reduces to $s(c, d)$.

2. The Eichler cohomology theorem.

Since $F_\nu(\tau + 1) = F_\nu(\tau)$ follows directly from the definition of F_ν (without regard to the nature of the $a_n(\nu)$), showing that $F_\nu(\tau)$ is a modular form of weight k on $\Gamma(1)$ reduces to proving the single transformation formula

$$\tau^{-k} F_\nu(-1/\tau) = F_\nu(\tau). \tag{31}$$

However, the presence of the subtracted polynomials $q(\tau; c, d)$ in (29) gives rise instead to a transformation formula of a more general kind, namely,

$$\tau^{-k} F_\nu(-1/\tau) = F_\nu(\tau) + p_\nu(\tau), \tag{32}$$

with $p_\nu(\tau)$ a polynomial "period" of degree $\le r = -k$.

In contrast to the case of $J(\tau)(k = 0$ and $\nu = 1)$, when $k < 0$ these polynomial periods do not always disappear. In fact, the identical vanishing of $p_\nu(\tau)$ depends upon the parameters k and ν, and in the generic situation $p_\nu(\tau)$ does not vanish. This, notwithstanding the fact that $p_\nu(\tau) \equiv 0$ for all $\nu \in Z^+$ when $k = 0, -2, -4, -6, -8$ and -12. Conspicuously, $k = 10$ does not belong on this list. (Verification of these facts is left aside.) Thus, while the exact formula (25) of Rademacher-Zuckerman shows that every modular form of weight $k < 0$ is a linear combination of the functions $F_\nu(\tau)$, the converse is not true (at least for $k = -10$ and even $k \le -14$). Of course, with fixed k one can form a linear combination of sufficiently many $F_\nu(\tau)$ to force the disappearance of the additive polynomial and thus obtain a modular form. There are stringent conditions upon such linear combinations, explained by Petersson through his "principal parts condition" [23, Theorems 1 and 3] and his "gap theorem" for automorphic forms [26, Theorem 1].

This state of affairs has made inevitable the frequent, prominent appearance of functions having transformation laws like (32) - that is, with additive polynomials - observable in the recent (since (1957)) study of modular and automorphic forms. These "Eichler integrals" - as they are called - are functions $F(\tau)$ holomorphic in $\mathcal{H}$ and satisfying

$$(\gamma\tau + \delta)^{-k} F(M\tau) = F(\tau) + p_M(\tau), \tag{33}$$

for all $M = \binom{* *}{\gamma \delta} \in \Gamma$, a discrete group of real linear fractional transformations. Here, k is an even integer ≤ 0, called the *weight* of the integral $F(\tau)$, and $p_M(\tau)$ is a polynomial in τ of degree at most $-k$. The $p_M(\tau)$ are the *period polynomials* of $F(\tau)$. Combining (32) and the fact that $F_\nu(\tau + 1) = F_\nu(\tau)$, we find that the functions $F_\nu(\tau)$ are Eichler integrals on $\Gamma(1)$, since $\tau \to \tau + 1$ and $\tau \to -1/\tau$ generate the group. Naturally, if $p_M(\tau) \equiv 0$ for all $M \in \Gamma$ in (33), then the Eichler integral F is really an automorphic form on Γ.

Of crucial importance in establishing a further nontrivial link between Eichler integrals and automorphic forms is the differentiation formula of G. Bol [2],

$$D^{(-k+1)}\{(\gamma\tau + \delta)^{-k} F(M\tau)\} = (\gamma\tau + \delta)^{k-2} F^{(-k+1)}(M\tau), \tag{34}$$

where $M = \binom{\alpha \beta}{\gamma \delta}$, with $\alpha\delta - \beta\gamma = 1$. Clearly, (34) implies that *the $(-k+1)^{st}$ derivative of an Eichler integral of weight k on Γ is an automorphic form on Γ of weight $2 - k$.* (34) follows for differentiable F by induction on $-k$ and for analytic F by the Cauchy integral formula.

An immediate consequence of (33) is the (cocycle) consistency condition

$$p_{M_1 M_2} = p_{M_1}|M_2 + p_{M_2}, \text{ for all } M_1, M_2 \in \Gamma, \tag{35}$$

where for convenience we have introduced the *slash operator*

$$(\phi|M)(\tau) = (\gamma\tau + \delta)^{-k}\phi(M\tau),$$

for $M = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}$. When ϕ is a polynomial of degree $\le -k$, so is $\phi|M$. A collection of polynomials $\{p_M | M \in \Gamma\}$ satisfying (35) - thus, necessarily of degree $\le -k$ - forms a (weight $-k$) *cocycle* on Γ. Given a fixed polynomial p of degree $\le -k$ it generates the cocycle $\{p_M | M \in \Gamma\}$ by means of $p_M = p|M - p$. We call a cocycle of this special form a *coboundary* and define the *Eichler cohomology group* $H^1_{-k}(\Gamma)$ as the quotient vector space of weight $-k$ cocycles modulo weight $-k$ coboundaries.

The identity (34) suggests a direct relationship between $H^1_{-k}(\Gamma)$ and automorphic forms of weight $2 - k$ on Γ, which we can establish as follows. Let G be such a form and F a $(-k + 1)$-fold anti-derivative of G. Then by (34) F satisfies (33), with $p_M(\tau)$ a polynomial of degree $\leq -k$ for each $M \in \Gamma$. This produces a mapping into $H^1_{-k}(\Gamma)$, if we attach to G the cohomology class in $H^1_{-k}(\Gamma)$ of the cocycle $\{p_M | M \in \Gamma\}$. Indeed, Eichler's classic paper [3], which initiated the study of $H^1_{-k}(\Gamma)$, identifies a distinguished subspace of $H^1_{-k}(\Gamma)$ with a direct sum of two spaces of automorphic forms of weight $2-k$ on Γ:

Eichler Cohomology Theorem. For $k \in Z, k \leq 0$, $H^1_{-k}(\Gamma)$ is isomorphic to the direct sum $C^+(\Gamma, 2-k) \oplus C^0(\Gamma, 2-k)$, provided Γ is an H-group [17, p. 266].

Remarks 1. $C^+(\Gamma, 2-k)$ is the (finite-dimensional) space of *entire* automorphic forms on Γ of weight $2-k$, those forms for which at each parabolic cusp of Γ the exponential expansion has no terms with negative exponents. $C^0(\Gamma, 2-k)$ is the subspace of *cusp forms* in $C^+(\Gamma, 2-k)$, those entire forms such that each expansion contains only terms with positive exponents.

2. For simplicity I have stated only a restricted form of the version of Eichler's theorem given in [6]. However, this form of the theorem exhibits the essence of the full result. Other versions include Eichler's original result [3] and [34, 4, 16, 18, 15].

The proof of the Eichler theorem given in [6] depends strongly upon Theorem 2, extended to general H-groups Γ. This generalization applies directly to establish a strong connection between the cocycle $\{p_M\}$ arising from F_ν and the cocycle $\{p^*_M\}$ arising from $F_{-\nu}$ (the result of replacing ν by $-\nu$ in the Fourier series definition of F_ν):

$$p^*_M(\tau) = \overline{p_M(\overline{\tau})}. \tag{36}$$

(See [10, (4.8)].) Then, for a linear mapping suitably defined from automorphic forms of weight $2-k$ into $H^1_{-k}(\Gamma)$, the relation (36) yields a proof that the mapping is one-to-one [6, pp. 570–571]. (This mapping necessarily keeps C^+ and C^0 disjoint, even though $C^0 \subset C^+$.) The proof that the range of this mapping consists of the entire space $H^1_{-k}(\Gamma)$ requires Petersson's generalized Riemann-Roch Theorem [27, Theorem 9].

The Eichler cohomology theorem may be regarded as stating that *every polynomial cocycle arises as the system of period polynomials of some Eichler integral*, and that *this Eichler integral is uniquely determined by the cohomology class of the given cocycle.* Like the Riemann-Roch theorem (more properly, Petersson's generalization of it), the Eichler cohomology theorem establishes a profound connection - only hinted at by (34) - between automorphic forms of weight $2-k (k \in Z, k \leq 0)$ and those of weight k. It shows that each entire automorphic form of weight $2-k$ gives rise to an "obstruction" to the existence of forms of weight k, and that each cusp form in fact gives rise to two such obstructions.

IV. Concluding Remarks

Although appearing eighteen years after [31], Eichler's work did not find its motivation in Rademacher's approach to $J(\tau)$. This is clear both from the internal evidence (Eichler's article itself) and from the fact that the necessary link is established not in Rademacher's work, but in the extensions of it to negative weight in [8, 10, 11], published between 1960 and 1962. That one can consider Eichler cohomology an outgrowth of Rademacher's work on $J(\tau)$ is an instance of hindsight, an example illustrating the familiar, yet striking, fact that developments which seem unrelated at first can turn out with time to be aspects of the same mathematical phenomenon.

I have not described all of the applications now in the literature of Rademacher's method. These include: (i) the use of the method to construct Poincaré series of weight 2 (in which case convergence problems arise in the definition (16)) [21, 35]; (ii) application to the construction of automorphic forms of real (not necessarily integral) nonpositive weights [22]. In [21] Lehner restricted his attention to Poincaré series of weight 2 on $\Gamma(1)$, while Smart [35] carried out a generalization to certain subgroups of finite index in $\Gamma(1)$. The work of Niebur [22], while significant principally for its extension of Rademacher's method to nonintegral weights, provdies new insights even for negative integral weight: when the weight is an integer the results of [22] do not reduce to those of [10], but strengthen them instead.

According to Paul Bateman, Rademacher tried without success to extend his method of [31] to nonintegral weights, in particular to the function $1/\eta(\tau)$, which has weight $-1/2$. We may therefore safely assume that, could he have known of it, Rademacher would have been most interested in Niebur's work. In the spring of 1963, well before that work was begun, I had the opportunity to tell Professor Rademacher about by own generalization to negative integral weights. As we walked alone, near the university campus in Madison, Wisconsin, I broached the subject and he siezed upon it with apparent interest. But, within moments, something distracted us; to my later regret, we never returned to the subject.

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