Hardy, Ramanujan's Work

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Lectures

by

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THE MATHEMATICAL WORK OF RAMANUJAN

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CONTENTS

MATHEMATICAL WORK OF RAMANUJAN

by

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The following topics will be considered in Professor Hardy's lectures, in whatever order is found most convenient:

Analytic Theory of Numbers
The Hardy-Ramanujan Theorem
Congruence properties of the partition function p(n)
$\tau(n)$
The Rogers-Ramanujan Theorem and Continued Fractions
The Dougall-Ramanujan Theorem
Asymptotic properties of p(n)

$$\int_0^\infty x^{s-1}\{\phi(0) - x\phi(1) + x^2\phi(2) - \cdots\}\,dx = \frac{\pi\,\phi(-s)}{\sin s\pi}$$

Reciprocal functions

The second topic will be given first, as it presupposes the least familiarity with the subjects involved.

1. The Hardy-Ramanujan Theorem

The number $1200 = 2^4 \cdot 3 \cdot 5^2$ is a "round number" in the decimal scale. Renouncing any especial allegiance to the decimal scale, we may characterize a round number as one which has a large number of prime factors in comparison with its size. Thus $2187 = 3^7$ is as "round" as 1200. It is a matter of common observation that most numbers are not round. It is a matter of interest to decide in what sense round numbers are rare. We may count the number of prime factors of any number in two ways. Let

$$n = p_1^{a_1} p_2^{a_2} \cdots p_v^{a_n}$$

be the decomposition of any number into prime factors. If we write

(1.1) $$f(n) = v, \quad F(n) = a_1 + a_2 + \cdots + a_v$$

then f(n) is the number of distinct primes dividing n, and F(n) is the number of prime factors of n, counting multiple factors multiply. It is obvious that f(n) is largest in comparison with n when

(1.2) $$n = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_r$$

and F(n) is largest when

(1.3) $$n = 2^r.$$

For (1.2) $f(n) = r$. In this case

(1.4) $$\log n = \sum_{p \le p_r} \log p$$

But in the classical theory of the distribution of primes [see "The Distribution of Prime Numbers" by A. E. Ingham, No. 30 in the Cambridge Tracts in Mathematics, pages 12-13] $\theta(x) = \sum_{p \le x} \log p$ and a result equivalent to the prime number theorem

(1.5) $$\pi(x) \sim \frac{x}{\log x}$$

is

(1.6) $$\theta(x) \sim x$$

Hence $\log n \sim p_r$. Another equivalent form of (1.5) is

(1.7) $$p_r \sim r \log r$$

Hence we have

$$\log n \sim r \log r$$

$$\log\log n \sim \log r + \log\log r \sim \log r$$

whence, dividing

(1.8)
$$\frac{\log n}{\log\log n} \sim r = f(n).$$

Thus as a universal theorem $f(n) = O(\frac{\log n}{\log\log n})$. Similarly from (1.3)

(1.9)
$$F(n) = r = \frac{\log n}{\log 2}$$

and universally $F(n) \leq \frac{\log n}{\log 2}$.

But in some sense, $f(n)$ and $F(n)$ are usually much smaller than these values. In general if $f(n)$ is an arithmetical function, which may be very irregular, and if

$$\sum_{n\leq x} f(n) \sim \sum_{n\leq x} \varphi(n)$$

where $\varphi(n)$ is some simple and regular function, it is natural to say that the average order of $f(n)$ is $\varphi(n)$. In this sense the average order of our $f(n) = v$ is easily shown to be $\log\log n$. For

$$\sum_{n\leq x} f(n) = \sum_{pm\leq x} 1$$

and summing first for p fixed we obtain

(1.10)
$$\sum_{n\leq x} f(n) = \sum_{p\leq x} [\frac{x}{p}] = x\sum_{p\leq x}\frac{1}{p} + O[\pi(x)].$$

Here there is an error of at most one in dropping each bracket and the number of brackets dropped is $\pi(x)$. But from Ingham, page 22

(1.11)
$$\sum_{p\leq x}\frac{1}{p} = \log\log x + B + O(\frac{1}{\log x})$$

Hence

(1.12)
$$\sum_{n\leq x} f(n) = x\log\log x + Bx + O(\frac{x}{\log x}) \sim x\log\log x$$

But we have also

$$\sum_{n\leq x} \log\log n \sim x\log\log x$$

The Hardy-Ramanujan Theorem, however, investigates a more sophisticated question. What is the normal number of prime factors of a number, that is the number for "most" integers? To give a precise meaning to "most" we define the concept of "almost all" numbers. We say that almost all numbers have a property P if $\overline{P}(x)$, the number of numbers less than x which do not have property P, is such that $\lim_{x\to\infty}\frac{\overline{P}(x)}{x} = 0$. Thus for example we may say that almost all numbers are composite, for the number of primes less than $x \sim \frac{x}{\log x}$ and $\lim_{x\to\infty}\frac{1}{x}\cdot\frac{x}{\log x} = 0$. In these terms we may state:

THE HARDY-RAMANUJAN THEOREM: The normal number of prime factors of an integer n is $\log\log n$, that is for each $\varepsilon > 0$, almost all numbers satisfy the relations

$$(1-\varepsilon)\log\log n < f(n) < (1+\varepsilon)\log\log n$$

and

$$(1-\varepsilon)\log\log n < F(n) < (1+\varepsilon)\log\log n.$$

There are two proofs of this theorem: we give a sketch of the original Hardy-Ramanujan proof and a more detailed account of the elegant but less revealing proof due to Turan.

We observe first that the order of magnitude of $f(n)$ and $F(n)$ must be the same. For

(1.13)
$$F(n) = \sum_{\substack{p,\mu \\ p^\mu | n}} 1$$

Hence

(1.14)
$$\sum_{n\leq x} F(n) = \sum_{p,\mu} [\frac{x}{p^\mu}]$$

From (1.10) and (1.14) we obtain

(1.15)
$$\sum_{n\leq x} [F(n) - f(n)] = \sum_{p} [\frac{x}{p^2}] + [\frac{x}{p^3}] + \dots$$

Hence

$$\sum_{n \leq x} [F(n) - f(n)] \doteq x\left[\sum_p \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right] = x \sum_p \frac{1}{p(p-1)}$$

Hence the <u>average value</u> of F(n) - f(n) is $\sum_p \frac{1}{p(p-1)} < 1$, and as F(n) - f(n) is always positive or zero, its <u>normal value</u> must be even less. More precisely, if X(n) is <u>any</u> function which tends to infinity with n, for almost all numbers F(n) - f(n) < X(n). Roughly we may say that F(n) - f(n) is almost always bounded.

THE HARDY-RAMANUJAN PROOF. If $\varpi_r(x)$ is the number of numbers $\leq x$ which have exactly r distinct prime factors

(1.16) $$\varpi_r(x) \sim \frac{x}{\log x} \frac{(\log \log x)^{r-1}}{(r-1)!}$$

[See Landau "Primzahlen", vol. 1, pp. 208. 211. Landau does not consider $\varpi_r(x)$ but the number of numbers $\leqq x$ for which F(n) = r.]

The derivation of

$$\varpi_r(x) \sim \frac{x}{\log x} \frac{(\log \log x)^{r-1}}{(r-1)!}$$

from the prime number theorem is quite elementary. For example, consider $\varpi_2(x)$

$$\varpi_2(x) = \sum_{pp' \leq x} 1 = \sum_{p \leq x} \pi\left(\frac{x}{p}\right) = \sum_{p \leq \frac{x}{2}} \pi\left(\frac{x}{p}\right)$$

since $\pi\left(\frac{x}{p}\right) = 0$ for $p > \frac{x}{2}$ Hence

$$\varpi_2(x) \sim \sum_{p \leq \frac{x}{2}} \frac{x}{p \log \frac{x}{p}} = \int_2^{\frac{x}{2}} \frac{x}{t \log \frac{x}{t}} \, d[\pi(t)]$$

Integrating by parts

$$\sum_{p \leq \frac{x}{2}} \frac{x}{p \log \frac{x}{p}} = \pi\left(\frac{x}{2}\right) \varphi\left(\frac{x}{2}\right) - \int_2^{\frac{x}{2}} \varphi'(t) \pi(t) \, dt$$

where

$$\varphi(t) = \varphi(x, t) = \frac{\frac{x}{t}}{\log\left(\frac{x}{t}\right)}$$

Hence

$$\sum_{p \leq \frac{x}{2}} \frac{x}{p \log \frac{x}{p}} = \pi\left(\frac{x}{2}\right) \frac{2}{\log 2} + \int_2^{\frac{x}{2}} \pi(t) \left(\frac{x}{t^2 \log \frac{x}{t}} - \frac{x}{t^2 \log^2 \frac{x}{t}}\right) dt$$

Here we need consider only the term of highest order

$$x \int_2^{\frac{x}{2}} \frac{\pi(t)\,dt}{t^2 \log \frac{x}{t}} \sim x \int_2^{\frac{x}{2}} \frac{dt}{t \log t \log \frac{x}{t}} \sim \frac{x}{\log x} \int_2^{\frac{x}{2}} \frac{dt}{t \log t} \sim \frac{x \log \log x}{\log x}$$

and so

$$\varpi_2(x) \sim \frac{x \log \log x}{\log x}$$

We have also

(1.17) $$[x] = \varpi_1(x) + \varpi_2(x) + \cdots$$

since on the right we have counted each integer less than x exactly once. Moreover we have the identity

(1.18) $$x = \frac{x}{\log x} e^{\log \log x} = \frac{x}{\log x}\left(1 + \xi + \frac{\xi^2}{2!} + \cdots\right)$$

where ξ = log log x. Observing that (1.16) gives a correspondence between terms of (1.17) and (1.18), we may reasonably expect that corresponding terms will to a certain extent be of the same order of magnitude. This similarity cannot, of course, be too close, since the first series terminates.

The largest term in (1.18) is that for which r = [ξ] + 1. We write

$$x = \frac{x}{\log x} \sum_1^\infty \frac{\xi^{r-1}}{(r-1)!} = \frac{x}{\log x} \sum \frac{\xi^{[\xi]+\mu-1}}{([\xi]+\mu-1)!}$$

where μ runs from -[ξ] + 1 to + ∞ By Sterling's formula

$$\frac{\xi^{[\xi]+\mu-1}}{([\xi]+\mu-1)!} \sim \frac{e^{\xi}}{\sqrt{2\pi\xi}}\, e^{-\frac{\mu^2}{2\xi}}$$

if μ is sufficiently small in comparison with ξ Hence (1.18) may be compared with

$$\frac{x}{\log x}\cdot \log x \cdot \frac{1}{\sqrt{2\pi\xi}} \sum e^{-\frac{\mu^2}{2\xi}} \quad \text{or with} \quad \frac{x}{\sqrt{2\pi\xi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\xi}}\,dt = x$$

and the part of this integral for which t is of higher order than $\sqrt{\xi}$ is negligible. Hence it is natural to suppose that practically all of the sum of the ϖ series comes from the terms in which $|r-\xi| = |r - \log\log x| = O(\sqrt{\xi}) = O(\sqrt{\log\log x})$. This argument can in fact be made rigorous by using inequalities for the $\varpi_r(x)$ instead of the asymptotic equalities. Actually the inequalities are less deep than the asymptotic equalities as they do not require the theory of complex variables, and in this sense the Hardy-Ramanujan proof is strictly elementary. The conclusion is that if $\chi(x)$ is any function of x such that $\frac{\chi(x)}{\sqrt{\log\log x}} \to \infty$ then almost all numbers not exceeding x have between $\log\log x \pm \chi(x)$ factors. Moreover almost all numbers n not exceeding x are in the interval $x^{\frac{1}{2}} < n < x$ and for this range $\log\log x = \log\log n + O(1)$. Hence formally:

If $\chi(n)$ is any function such that $\lim_{n\to\infty} \chi(n)/\sqrt{\log\log n} = \infty$ then almost all numbers have between $\log\log n \pm \chi(n)$ prime factors, whether multiple factors are counted multiply or not.

There is an interesting corollary about d(n), the number of divisors of n. It is known that $d(1) + d(2) + \dots + d(n) \sim n \log n$, whence the average order of d(n) is log n. But the Hardy-Ramanujan Theorem may be used to show that this is not the normal order of d(n).

If, as before $n = p_1^{a_1} \dots p_v^{a_v}$, $d(n) = \prod_{i=1}^{v}(1+a_i)$. Now as a_i is a positive integer: $a \leq 1 + a_i \leq 2^{a_i}$. Multiplying these inequalities for $i = 1, \dots v$

$$2^{v} \leq \prod(1+a_i) \leq 2^{\sum a_i} \quad \text{or} \quad 2^{f(n)} \leq d(n) \leq 2^{F(n)}$$

But by the Hardy-Ramanujan Theorem, both f(n) and F(n) are normally log log n. Hence d(n) is normally about $2^{\log\log n} = (\log n)^{\log 2} = \log n^{.69315}$ which is of a considerably lower order than log n, the average value of d(n). This means that the numbers with an abnormally large number of divisors have such an exceedingly large number of divisors that they dominate the average of d(n). The irregularities of f(n) and F(n) are not great enough to produce a similar effect.

TURAN'S PROOF OF THE HARDY-RAMANUJAN THEOREM. We use not only

(1.19)
$$\sum_{n\leq x} f(n) = \sum_{p\leq x}\left[\frac{x}{p}\right] \sim x\sum_{p\leq x}\frac{1}{p} = x\log\log x + O(x)$$

previously mentioned, but also

(1.20)
$$\sum_{n\leq x}\{f(n)\}^2 = \sum_{\substack{pp'\leq x\\ p\neq p'}}\left[\frac{x}{pp'}\right] + \sum_{p\leq x}\left[\frac{x}{p}\right]$$

For

$$\sum_{n\leq x}\{f(n)\}^2 = \sum_{n\leq x}\Big(\sum_{p|n} 1\cdot\sum_{p'|n} 1\Big) = \sum_{p\mu=p'\mu'\leq x} 1 = \sum_{\substack{p\neq p'\\ pp'\mu\leq x}} 1 + \sum_{p\mu\leq x} 1 = \sum_{\substack{p\neq p'\\ pp'\leq x}}\left[\frac{x}{pp'}\right] + \sum_{p\leq x}\left[\frac{x}{p}\right]$$

by summing first with respect to μ and m. From (1.20)

(1.21)
$$\sum_{n\leq x}\{f(n)\}^2 = x\sum_{pp'\leq x}\frac{1}{pp'} + O(x)$$

Also

(1.22)
$$\Big(\sum_{p\leq x}\frac{1}{p}\Big)^2 \leq \sum_{pp'\leq x}\frac{1}{pp'} \leq \Big(\sum_{p\leq x}\frac{1}{p}\Big)^2$$

so that

(1.23) $$\sum_{pp'\le x}\frac{1}{pp'} = \{\log\log x + O(1)\}^2 = (\log\log x)^2 + O(\log\log x)$$

Hence if we write ξ again for log log x, we have, using (1.19), (1.21) and (1.23)

(1.24) $$\begin{aligned}\sum_{n\le x}\{f(n)-\xi\}^2 &= \sum_{n\le x}\{f(n)\}^2 - 2\xi\sum_{n\le x} f(n) + \xi^2[x] \\ &= x\{\xi^2+O(\xi)\} - 2\xi x\{\xi+O(1)\} + \xi^2\{x+O(1)\} \\ &= O(\xi x)\end{aligned}$$

But this would be impossible if $f(n) - \xi$ were of order higher than $\sqrt{\xi}$ for a finite proportion of the n. Hence (after what was said before) we have the same result for F(n).

A natural question supplementary to the Hardy-Ramanujan theorem is: How many numbers less than x are there with exactly ξ = [log log x] prime factors? From $\varpi_\nu(x) \sim \frac{x}{\log x}\frac{\xi^\nu}{\nu!}$ we may, by use of Stirling's theorem obtain $\varpi_\xi(x) < \frac{Ax}{\sqrt{\xi}}$ It seems highly plausible that this is the exact order of $\varpi_\xi(x)$, but the best that has been proved is $\frac{Ax}{\xi^{\frac{\xi}{2}}} < \varpi_\xi(x)$

2. Analytic Theory of Numbers

The following theorems on the analytic theory of numbers were contained in letters to Professor Hardy written by Ramanujan early in 1913 while he was in his own words "a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 23 years of age."

(1) $\pi(x) = \int_2^x \frac{dt}{\log t} + \rho(x)$

$\rho(e^{2\pi x})$ is very small when x lies between 0 and 3 (its value is less than a few hundreds when x = 3 and rapidly increases when $x > 3$.)

(2) $\pi(x) = \int_\mu^x \frac{dt}{\log t} - \frac{1}{2}\int_\mu^{\sqrt{x}}\frac{dt}{\log t} - \frac{1}{3}\int_\mu^{\sqrt[3]{x}}\frac{dt}{\log t} \cdots\cdots$

(3) statements about the best way to calculate $\int\frac{dt}{\log t}$ and $\pi(x)$ from (2).

(4) The number of prime numbers less than $e^{a'}$ is $\int_0^\infty \frac{a^x\,dx}{x\zeta(x+1)\Gamma(x+1)}$

(5) The number of prime numbers less than n is $\frac{2}{\pi}\left\{\frac{2}{B_2}\frac{\log n}{2\pi} + \frac{4}{3B_4}\left(\frac{\log n}{2\pi}\right)^3 + \frac{6}{5B_6}\left(\frac{\log n}{2\pi}\right)^5 + \cdots\right\}$

(6) The difference between the number of prime numbers 4n-1 less than x and those 4n+1 tends to infinity.

(7) Corresponding statements about other arithmetic progression.

(8) The number of numbers $2^p3^q < n$ is $\frac{1}{2}\frac{\log 2n \log 3n}{\log 2 \log 3}$

(9) ν a number $\mu(\nu) = -1$ (odd number of different prime factors)

(a) The number of $\nu \le n$ is $\frac{3n}{\pi^2}$

(b) $\sum\frac{1}{\nu^2} = \frac{9}{2\pi^2}$

(10) $d(1) + \ldots + d(n) = n\log n + (2\gamma - 1)n + \frac{1}{2}d(n) + \ldots$

(11) The number of numbers between A and x which are either squares or sums of two squares is

$K\int_A^x \frac{dt}{\sqrt{\log t}} + \theta(x)$ where K = 0,764... and (x) is very small compared with the previous integral.

In Ramanujan's first formula

(2.1) $$\pi(x) = \int_2^x \frac{dt}{\log t} + \rho(x)$$

he must have meant to write $-\rho(x)$ since his second formula

(2.2) $$\pi(x) = \int_{\mu}^{x} \frac{dt}{\log t} - \frac{1}{2}\int_{\mu}^{\sqrt{x}} \frac{dt}{\log t} - \frac{1}{3}\int_{\mu}^{\sqrt[3]{x}} \frac{dt}{\log t} \cdots\cdots$$

certainly implies that $\pi(x)$ is less than

(2.3) $$li\, x = \int_{2}^{x} \frac{dt}{\log t}$$

This second formula has, in calculation, shown a remarkable agreement with tabulations of $\pi(x)$; in fac the agreement is much too good to be justified theoretically.

In the following table x and $\pi(x)$ are given in the first two columns (the number one is not counted as a prime). The last three columns give the errors (differences from $\pi(x)$) for the formulae (2.2) of Riemann-Ramanujan, (2.3) of Chebyshev, and that of Legendre

(2.4) $$\pi(x) = \frac{x}{\log x - 1.08}$$

x	$\pi(x)$	Errors: Riemann Ramanujan	Errors: Chebyshev	Errors: Legendre
100,000	9,592	-5	+38	-4
1,000,000	78,498	+30	+130	+45
2,000,000	148,933	-9	122	+43
3,000,000	216,816	0	155	97
4,000,000	283,146	+33	206	177
5,000,000	348,513	-64	125	131
6,000,000	412,849	+24	228	272
7,000,000	476,648	-38	179	264
8,000,000	539,777	-6	223	351
9,000,000	602,489	-53	187	361
10,000,000	664,579	+88	339	561
100,000,000	5,761,455		754	
1,000,000,000	50,847,478		1757	
	Formulae:	$\sum \frac{\mu(m)}{m} li\, x^{\frac{1}{m}}$	$li\, x = \int_{2}^{x} \frac{dt}{\log t}$	$\frac{x}{\log x - 1.08}$

Littlewood in 1914 showed that $\rho(x)$ assumes, for large x, values of order at least

(2.5) $$\frac{\sqrt{x}}{\log x} \log\log\log x$$

and of either sign. Hence

(2.6) $$\pi(x) - li\, x$$

changes sign for x beyong all limit, and assumes values, of either sign, of an order higher than all the terms of the series after the first.

These values of x, however, seem to be enormous, the most known being that

(2.7) $$\pi(x) > li\, x$$

for some

(2.8) $$x < 10^{10^{10^{34}}}$$

It is not surprising that no computations indicate the changes of sign.

The inequality

(2.9) $$\pi(x) < li\, x$$

can be reestablished if we interpret it in an average sense. If the Riemann Hypothesis is true, then

(2.10)
$$\frac{1}{x}\int_2^x (\pi(t) - \mathrm{li}\, t)\, dt \to -\infty$$

(and indeed the truth of the Riemann Hypothesis is necessary and sufficient for this).

It is known that, if we define li x by

(2.11)
$$\mathrm{li}\, x = \int_0^x \frac{dt}{\log t}$$

(the integral being a Cauchy principal value when $x > 1$) then

(2.12)
$$\mathrm{li}\, x = \gamma + \log\log x + \sum_1^\infty \frac{(\log x)^m}{m \cdot m!}$$

Hence

(2.13)
$$\mathrm{li}\, x^{\frac{1}{n}} = \gamma + \log\log x - \log n + \sum_1^\infty \frac{(\log x)^m}{m \cdot m!\, n^m} = \gamma + \log\log x - \log n + O\left(\frac{1}{n}\right)$$

for large n. For the convergence of

(2.14)
$$R(x) = \sum_1^\infty \frac{\mu(n)}{n}\, \mathrm{li}\, x^{\frac{1}{n}}$$

it is necessary and sufficient that $\sum \frac{\mu(n)}{n} \log n$ be convergent. In fact it is known that

(2.15)
$$\sum_1^\infty \frac{\mu(n)}{n} = 0 \qquad \sum_1^\infty \frac{\mu(n)}{n} \log n = -1$$

The convergence even of the first series is as deep as the prime number theorem; but the results are plausible because

(2.16)
$$\sum \frac{\mu(n)}{n^{1+\delta}} = \frac{1}{\zeta(1+\delta)} \to 0 \qquad \sum \frac{\mu(n)\log n}{n} = \frac{\zeta'(1+\delta)}{\zeta(1+\delta)} \to -1$$

when $\delta \to 0$.

If we substitute from (2.11) into (2.14) and invert the order of summation, we obtain

(2.17)
$$R(x) = 1 + \sum_1^\infty \sum_1^\infty \frac{(\log x)^m \mu(n)}{m \cdot m!\, n^{1+m}} = 1 + \sum_1^\infty \frac{(\log x)^m}{m \cdot m!\, \zeta(m+1)} = g(x)$$

say. This transformation is due to Gram.

Ramanujan's series (5)

(2.18)
$$G(x) = \frac{2}{\pi}\left\{\frac{2}{B_2}\frac{\log x}{2\pi} + \frac{4}{3B_4}\left(\frac{\log x}{2\pi}\right)^3 + \frac{6}{5B_6}\left(\frac{\log x}{2\pi}\right)^5 + \cdots\right\}$$

is twice the odd terms in g(x): i.e.

(2.19)
$$G(x) = g(x) - g\left(\frac{1}{x}\right)$$

It can be shown that $g(\frac{1}{x}) \to 0$ so that g(x), G(x), and R(x) agree with error o(1).

It can also be proved that

(2.20)
$$J(x) = \int_0^\infty \frac{x^t\, dt}{t\, \zeta(t+1)\, \Gamma(t+1)} = R(x) + o(1)$$

so that Ramanujan's integral (which was new) agrees to the same extent.

3. The Development and Present State of Knowledge of the Theory of the Distribution of Primes

The "prime number theorem"

(3.1)
$$\pi(x) \sim \frac{x}{\log x}$$

appears as a conjecture in Legendre and Gauss, but neither was explicit as to the exactitude implied by the statement. Gauss even mentioned li x, but without indicating whether or not he considered this a better approximation than $\frac{x}{\log x}$.

Equivalent to the prime number theorem is

(3.2)
$$p_n \sim n \log n$$

The identity $\prod \left(\frac{1}{1-\frac{1}{p^s}}\right) = \sum \frac{1}{n^s}$ $s > 1$ makes it natural to suppose that $\prod\left(\frac{1}{1-\frac{1}{p}}\right)$ and $\sum \frac{1}{n}$ diverge similarly. As the difference between $\sum \log \frac{1}{1-\frac{1}{p}}$ and $\sum \frac{1}{p}$ is a convergent series, we may expect that $\sum_{p \le x} \frac{1}{p} \sim \log\left(\sum_{n \le x} \frac{1}{n}\right) \sim$ log log x, and this suggests $p_n \sim n \log n$ since

$$\sum \frac{1}{n \log n} \sim \log \log x$$

Again the identity

(3.3) $$x! = \prod_{p \le x} \left(p^{\left[\frac{x}{p}\right] + \left[\frac{x}{p^2}\right] + \cdots} \right)$$

gives $\log x! = \sum_{p \le x} \log p \left\{ \left[\frac{x}{p}\right] + \left[\frac{x}{p^2}\right] + \ldots \right\}$ and using Stirling's Theorem $x \log x = x \sum_{p \le x} \frac{\log p}{p} + O\left(\sum_{p \le x} \log p \right)$. If we assume the weak statement $\pi(x) = o(x)$, then

$$x \log x \sim x \sum_{p \le x} \frac{\log p}{p} \qquad \log x \sim \sum_{p \le x} \frac{\log p}{p}$$

and here again we are led to expect $p_n \sim n \log n$.

The first definite progress was made when Chebyshev proved in 1848 that

(3.4) $$\frac{Ax}{\log x} < \pi(x) < \frac{Ax}{\log x}$$

and $An \log n < p_n < An \log n$.

It is more convenient to consider, instead of $\pi(x)$

(3.5) $$\theta(x) = \sum_{p \le x} \log p \qquad \psi(x) = \sum_{p^m \le x} \log p$$

These forms are really more natural arithmetically since they are concerned with the multiplication of primes rather than their enumeration, and $\psi(x)$ presents itself naturally analytically. These two functions are connected by the relation

(3.6) $$\psi(x) = \theta(x) + \theta(x^{\frac{1}{2}}) + \theta(x^{\frac{1}{3}}) + \ldots .$$

In terms of the ζ function

(3.7) $$\zeta(s) = \prod \frac{1}{1 - p^{-s}}$$

(3.8) $$\log \zeta(s) = \sum \log \frac{1}{1 - p^{-s}} = \sum_{p, m} \frac{1}{m p^{ms}}$$

(3.9) $$-\frac{\zeta'(s)}{\zeta(s)} = \sum \frac{\log p}{p^{ms}} = \sum \frac{\Lambda(n)}{n^s}$$

where $\Lambda(n) = \log p$ for $n = p^m$

$= 0$ otherwise.

Hence $\psi(x) = \sum_{p^m \le x} \log p = \sum_{n \le x} \Lambda(n)$. Chebyshev proved that $Ax < \frac{\theta}{\psi} < Ax$, and showed that the prime number theorem is equivalent to $\theta \sim x$ or $\psi \sim x$. Even more, he showed that, considering the three ratios $\frac{\pi(x)}{\frac{x}{\log x}}$, $\frac{\theta(x)}{x}$, $\frac{\psi(x)}{x}$, if any one tends to a limit, then all do and the limit must be unity [See the first chapter of Ingham for proofs.]

The function asymptotic to $\mathrm{li}\, x$ which arises most naturally in the theory is not $\pi(x)$, but a function which Ingham denotes by $\Pi(x)$.

(3.10) $$\Pi(x) = \sum_{p^m \le x} \frac{1}{m}$$

This arises from the identity

(3.11) $$\log \zeta(s) = \sum \frac{1}{m p^{ms}} = \sum a_n n^{-s}$$

where $a_n = \frac{1}{m}$ for $n = p^m$

$= 0$ otherwise

and hence $$\Pi(x) = \sum_{n \le x} a_n$$

We do have however

(3.12) $$\Pi(x) = \pi(x) + \tfrac{1}{2} \pi(x^{\frac{1}{2}}) + \tfrac{1}{3} \pi(x^{\frac{1}{3}}) \ldots$$

The natural associations in the analytic theory are

$\psi(x)$ with x

$\pi(x)$ with $\text{li}\, x = \int_2^x \frac{dt}{\log t}$

$\pi(x)$ with the Riemann-Ramanujan series $\sum \frac{\mu(m)}{m} \, li \, x^{\frac{1}{m}}$

The modern methods that have been applied to the theory of distribution of primes are "transcendental" in that they involve the representation of $\psi(x)$ by Cauchy's theorem, and estimations are made by an appropriate change of the contour of integration.

As in (3.9) $\frac{\zeta'(s)}{\zeta(s)} = \sum \Lambda(n) n^{-s}$ Let $s = \sigma + it$ and let $f(s)$ be a function represented by a Dirichlet series

(3.13) $$f(s) = \sum a_n n^{-s}$$

absolutely convergent for $\sigma > 1$; and let

(3.14) $$A^*(x) = \frac{1}{2\pi i}\int f(s)\frac{x^s}{s}\, ds$$

Then

(3.15) $$A^*(x) = \sum_{n \le x}{}' a_n$$

where the * means that if x is integral a_x is to be taken with the coefficient $\frac{1}{2}$. This yields the normal value at a discontinuity $f(x) = \frac{1}{2}[f(x+0) + f(x-0)]$. Now

(3.16) $$\psi(x) = \sum_{n \le x} \Lambda(n) = \frac{-1}{2\pi i}\int \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\, ds$$

where the contour of integration is taken upwards along a straight line parallel to the imaginary axis to the right of $\sigma = 1$. The function $\zeta(s)$ has a simple pole $\frac{1}{s-1}$ at $s = 1$ and no other singularity in the finite part of the plane. $\zeta(s)$ has "trivial zeros" for $s = -2, -4, \ldots$. Hence $\frac{\zeta'(s)}{\zeta(s)}$ has poles $\frac{1}{s-1}, -\frac{1}{s-2n}, -\frac{1}{s-\rho}$ where the ρ are the "non-trivial zeros" and are all located in the strip $0 \le \Re(\rho) \le 1$ and are symmetric about the line $\sigma = \frac{1}{2}$. The "Riemann Hypothesis" (still unproved or disproved) is that all have $\Re(\rho) = \frac{1}{2}$.

To get a first approximation to the proof of the prime number theorem let us assume that the Riemann Hypothesis is true. It is natural to suppose that we may move the contour of integration across $\sigma = 1$, and allowing for the pole at $s = 1$, (3.16) becomes

(3.17) $$\psi(x) = x + \frac{1}{2\pi i}\int \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\, ds$$

where the contour is now a line parallel to the imaginary axis passing between $\sigma = \frac{1}{2}$ and $\sigma = 1$. We move the contour as far to the left as possible without crossing any singularities of the integrand, on the Riemann Hypothesis to $\sigma = \beta$ where β is any real number greater than $\frac{1}{2}$, and it may be expected that the integral is $O(x^\beta)$. If this is so

(3.18) $$\psi(x) = x + O(x^\beta) \qquad \text{any } \beta > \tfrac{1}{2}.$$

This argument is not valid as it stands since the integral is not absolutely convergent. We shall see later how this difficulty may be avoided by the use of "Tauberian Theorems".

If we have proved (3.18), then it is easy to obtain the corresponding estimates of $\Pi(x)$ and $\pi(x)$. In fact

(3.19) $$\Pi(x) = \sum_{p^m \le x} \frac{1}{m} = \sum_2^x \frac{\Lambda(n)}{\log n}$$

Hence

(3.20) $$\Pi(x) = \int_2^x \frac{d\,\psi(t)}{\log t} = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t\,(\log t)^2}\, dt$$

Again

(3.21) $$li\, x = \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + \int_2^x \frac{t}{t\,(\log t)^2}\, dt$$

and subtracting (3.21) from (3.20),

(3.22) $$\pi(x) - li\, x = \frac{\psi(x)-x}{\log x} + \int_2^x \frac{\psi(t)-t}{t(\log t)^2}\,dt = O(x^\beta)$$

The Möbius inversion formula states that if $f(y)$ and $F(y)$ are two functions such that

(3.23) $$F(y) = f(y) + f(2y) + f(3y) + \dots$$

then

(3.24) $$f(y) = F(y) - F(2y) - F(3y) - \dots = \sum \mu(n)F(ny).$$

Applying this to (3.12) we obtain

(3.25) $$\pi(x) = \Pi(x) - \tfrac{1}{2}\Pi(x^{\frac{1}{2}}) - \tfrac{1}{3}\Pi(x^{\frac{1}{3}}) - \dots = \sum \frac{\mu(m)}{m}\Pi(x^{\frac{1}{m}})$$

whence, from (3.22),

(3.26) $$\pi(x) = li\ x + O(x^\beta).$$

It is in fact known that $\psi(x) - x$ cannot be of smaller order than $\sqrt{x}$. The difference between $\Pi(x)$ and $\pi(x)$ is not of higher order than $\pi(x^{\frac{1}{2}})$, or than $\frac{\sqrt{x}}{\log x}$. Hence, so far as formulae of this type are concerned, the terms $-\frac{1}{2}\ li\ (x^{\frac{1}{2}}) - \frac{1}{3}\ li\ (x^{\frac{1}{3}}) \dots$ cannot play an important rôle and we may ignore them.

Without the use of the unproved Riemann Hypothesis, the best that could be proved until recently is

(3.27) $$\psi(x) = x + O(x\, e^{-A\sqrt{\log x}})$$

Littlewood and Vinogradoff have gone just a little further, but their analysis is very difficult.

There is another way in which we may treat the integral in (3.16). We might deform the contour to $-\infty$ and obtain the explicit formula

(3.28) $$\psi_0(x) = x - \sum \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)$$

Here $\psi_0(x)$ is the same as $\psi(x)$ except that it has standard discontinuities:

$$\psi_0(x) = \tfrac{1}{2}[\psi_0(x+0) + \psi_0(x-0)]$$

This identity is substantially equivalent to one for $\pi(x)$ found without a rigorous proof by Riemann.

4. A Second Approximation to the Proof of the Prime Number Theorem

Let $f(s)$ be a function represented by a Dirichlet series

(4.1) $$f(s) = \sum a_n n^{-s}$$

where $f(s)$ has certain general characteristics known to be possessed by the specific functions of interest in the theory of distribution of primes, videlicet:

(4.1) is an absolutely convergent series for $\sigma > 1$; $f(s)$ is regular for $\sigma \geq 1$ except for a pole $\frac{1}{s-1}$.

Let us now suppose that the coefficients a_n are real and bounded negatively, $a_n > -K$, and also that $f(\sigma + i t) = O(|t|^\alpha)$ $0 < \alpha < 1$.

Write

(4.2) $$A(x) = \sum_{n \leq x}{}' a_n$$

where $\sum'$ means that if x is integral a_x is to be taken with a coefficient $\frac{1}{2}$. Then let

(4.3) $$A_1(x) = \int_0^x A(u)\,du$$

Since

(4.4) $$A(x) = \frac{1}{2\pi i}\int_C f(s)\frac{x^s}{s}\,ds$$

it follows that

(4.5) $$A_1(x) = \frac{1}{2\pi i}\int_C f(s)\frac{x^{s+1}}{s(s+1)}\,ds$$

In this way we also obtain

(4.6) $$x + O(1) = \int_C \zeta(s)\frac{x^s}{s}\,ds$$

and

(4.7) $$\frac{x^2}{2} + O(x) = \int_C \zeta(s) \frac{x^{s+1}}{s(s+1)}\, ds$$

Subtracting (4.7) from (4.5)

(4.8) $$A_1(s) - \tfrac{1}{2}x^2 = \int_{\sigma=1} (f - \zeta) \frac{x^{s-1}}{s(s+1)}\, ds + O(x^2) = x^2 \int F(t) e^{it \log x}\, dt$$

where $\int_{-\infty}^{+\infty} |F(t)|\, dt$ is convergent.

Now by the Riemann-Lebesgue theorem for trigonometric integrals, the right-hand side of this equation is $O(x^2)$. Hence we have

(4.9) $$A_1(x) \sim \tfrac{1}{2}x^2$$

But this is not quite our goal, as what we wish to prove is

(4.10) $$A(x) \sim x$$

It is at this stage of the argument that the Tauberian element enters. Every known proof of the prime number theorem involves two essentially distinct parts, one function-theoretical, involving either contour integration or Fourier transforms, and the other a Tauberian theorem. In general the weaker the function theoretical part of the argument, the stronger the Tauberian theorem must be, and conversely. Here the Tauberian element is simple. The theorem which we require is:

THEOREM 4.1. Suppose that $f(x) \sim \tfrac{1}{2}Cx^2$ and that $f'(x) + Kx$ is, for some K an increasing function of x. Then $f'(x) \sim Cx$.

To prove the theorem we may suppose C = 0, that is we replace $f(x)$ by $f(x) - \tfrac{1}{2}Cx^2$. In short, we must prove that if $f(x) = o(x^2)$ and if $f'(x) + Kx$ is increasing, then $f'(x) = o(x)$. In our case $f(x)$ is $A_1(x)$ and $f'(x)$ is $A(x)$ and K = 0.

Assume the theorem false. Then either

(a) $f' > \delta x_i$ for x_i beyond all limit; or (b) $f' < -\delta x_i$ for x_i beyond all limit.

In both cases we are led to a contradiction. The proof is similar for both cases and we shall give that for (a). Assume

(4.11) $$f'(x_i) > \delta x_i \qquad x_i \text{ beyond all limit}$$
$$f'(x) + Kx \text{ increases}$$

Then for $x_r < x < x_r + \frac{\delta}{2K} x_r$,

$$f'(x) + Kx \geq f'(x_r) + Kx_r$$
$$f'(x) \geq \delta x_r - K(x - x_r) > \tfrac{1}{2}\delta x_r$$

But this contradicts the assumption $f(x) = o(x^2)$ for

(4.12) $$f\left(x_r + \frac{\delta}{2K} x_r\right) - f(x) = \int_{x_r}^{x_r + \frac{\delta}{2K}x_r} f'(x)\,dx > \tfrac{1}{2}\delta x_r \frac{\delta x_r}{2K} = \frac{\delta}{4K} x_r^2$$

and the difference between two values of a function which is $o(x^2)$ must be $o(x^2)$.

It would be very easy to deduce the prime number theorem from false Tauberian theorems. There is an Abelian theorem that if $s_n = a_1 + a_2 + \dots + a_n \sim An$, then $f(s) = \sum \frac{a_n}{n^s} \sim \frac{A}{s-1}$ as $s \to 1$. If there were a Tauberian theorem which stated that $f(s) \sim \frac{A}{s-1}$ as $s \to 1$ implies $s_n \sim An$ if $a_n \geq 0$, then we could easily prove the prime number theorem in the following way: $\sum \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} \sim \frac{1}{s-1}$ as $s \to 1$ and hence $\psi(n) = s_n \sim n$, which is the prime number theorem. But it may be taken as axiomatic that any theorem which proves the prime number theorem easily must be false. In this case $a_n = n$ for $n = 2^m$, $a_n = 0$ otherwise, is an example showing the falsity of the supposed Tauberian theorem.

In our applications

(4.13) $$f(s) = -\frac{\zeta'(s)}{\ }$$

and we have to know that

(4.14) $$\left|\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right| = O|t|^{\alpha}$$

for $\sigma \geq 1$. Obviously this implies in particular that

(4.15) $$\zeta(1+it) \neq 0$$

This last proposition is essential in all proofs of the prime number theorem.

There is a deep Tauberian theorem which may be used to prove the prime number theorem:

THEOREM 4.2. If $\sum a_n e^{-ny} \sim \frac{C}{y}$ as $y \to 0$ and $a_n > 0$, then $A(x) \sim Cx$.

For we need only take $a_n = \Lambda(n)$. Here we can replace (4.14) by

(4.16) $$|f(\sigma + it)| = O(e^{A|t|})$$

for some A. Wiener's work shows that in fact no such O condition is necessary, and so enables us to simplify the function theoretic part of the proof a great deal (naturally at the expense of the Tauberian part).

There is a proof of the prime number theorem intermediate between the Hadamard proof and Wiener's (Hardy and Littlewood, Acta Mathematica 1916). In this it is shown first that

(4.17) $$\sum \Lambda(n) e^{-ny} \sim \frac{1}{y}$$

and the prime number theorem then follows by the Tauberian Theorem 4.2.

5. Ramanujan's Argument for the Truth of the Prime Number Theorem

Ramanujan's argument has been altered in the following in order to make it conform as much as possible to the true state of affairs. It must be remembered, however, that he was not looking for an asymptotic formula for $\pi(K)$ but for an exact formula or at least one with a bounded error term.

Write

(5.1) $$\varphi(y) = \sum_p \log p \sum_{m=1}^{\infty} e^{-p^m y} - \log 2 \sum_{m=1}^{\infty} 2^m e^{-2^m y} = \varphi_1(y) - \varphi_2(y)$$

Now consider

$$\Phi(y) = \varphi(y) - \varphi(2y) + \varphi(3y) - \varphi(4y) \cdots\cdots$$

(A) $$\Phi_1(y) = \sum_p \log p \sum_m \frac{1}{e^{p^m y}+1} = \sum \frac{\Lambda(n) e^{-ny}}{1+e^{-ny}}$$

$$\Phi_2(y) = \log 2 \sum \frac{2^m}{e^{2^m y}+1} = \frac{2\log 2}{e^{2y}-1}$$

(B) $$\Phi_1(y) = e^{-y}\log 1 - e^{-2y}\log 2 + e^{-3y}\log 3 \cdots\cdots\cdots + \frac{2\log 2}{e^{2y}-1}$$

To verify formula (B) note that

(5.1) $$\Phi_1(y) = \varphi_1(y) - \varphi_1(2y) + \varphi_1(3y) - \cdots\cdots$$

Now write

(5.2) $$\psi_1(y) = \sum \frac{\Lambda(n) e^{-ny}}{1-e^{-ny}} = \sum_{r,n=1}^{\infty} \Lambda(n) e^{-rn} = \sum c_m e^{-my}$$

Here $\psi_1(y)$ differs from $\Phi_1(y)$ only by a sign in the denominator.

(5.3) $$c_m = \sum_{rn=m} \Lambda(n) \text{ or } \quad c_m = \sum_{n|m} \Lambda(n)$$

Hence if $m = \prod p^a$

(5.4) $$c_m = \sum_{p^\mu|m} \log p = \sum_{p|m} a \log p = \sum_{p|m} \log p^a = \log m$$

Consequently

(5.5) $$\psi_1(y) = \sum_{m=1}^{\infty} \log m \cdot e^{-my}$$

Replacing y by 2y in this formula

(5.6) $$\sum \frac{\Lambda(n)\, e^{-2ny}}{1-e^{-2ny}} = \sum \log 2m \cdot e^{-2my} - \log 2 \sum e^{-2my} = \sum \log 2m\, e^{-2my} - \frac{\log 2 \cdot e^{-2y}}{1-e^{-2y}}$$

Subtracting twice this from $\psi_1(y)$ we obtain

(5.7) $$\begin{aligned} \psi_1(y) - 2\psi_1(2y) &= \sum \log m\, e^{-my} - 2\left(\sum \log 2m\, e^{-2my} - \frac{\log 2 \cdot e^{-2y}}{1-e^{-2y}}\right) \\ &= e^{-y}\log 1 - e^{-2y}\log 2 + e^{-3y}\log 3 \cdots\cdots + \frac{2\log 2 \cdot e^{-2y}}{1-e^{-2y}} \end{aligned}$$

But on the other hand

(5.8) $$\begin{aligned} \psi_1(y) - 2\psi_1(2y) &= \sum \frac{\Lambda(n)\, e^{-ny}}{1-e^{-ny}} - 2\sum \frac{\Lambda(n)\, e^{-2ny}}{1-e^{-2ny}} = \sum \frac{\Lambda(n)\left[e^{-ny} + e^{-2ny} - 2e^{-2ny}\right]}{1-e^{-2ny}} \\ &= \sum \frac{\Lambda(n)\left[e^{-ny} - e^{-2ny}\right]}{1-e^{-2ny}} = \sum \frac{\Lambda(n)\, e^{-ny}}{1+e^{-ny}} \\ &= \Phi_1(y) \end{aligned}$$

Hence, combining (A) and (B) we have

(C) $\quad \Phi(y) = e^{-y}\log 1 - e^{-2y}\log 2 \cdots\cdots$

Let us assume (and it is in fact true) that

(D) $\quad \Phi(y) \longrightarrow \text{limit} \quad \text{as } y \to 0$

For the series log 1 - log 2 + log 3 ... is summable $(C, 1)$. Our assumption is that this series is Abel summable, and every series which is summable $(C, 1)$ is also Abel summable. We rewrite (D) as

(E) $\quad \varphi(y) - \varphi(2y) + \varphi(3y) - \varphi(4y) \cdots \cdots \longrightarrow \ell$

where $\quad \varphi = \varphi_1 - \varphi_2 \qquad \varphi_1 = \sum \Lambda(n)\, e^{-ny} \qquad \varphi_2 = \log 2 \sum 2^m e^{-2^m y}$

To this point Ramanujan's work is correct. But here he asserts that (E) implies that $\varphi(y) \to$ limit, and this is in fact not true. A correct form of what Ramanujan wishes to use is

(F) $\quad \varphi_1(y) \sim \frac{1}{y} \quad \text{as } y \to 0$

and this relation is true. As remarked above, (F) plus a Tauberian theorem yields the prime number theorem. But Ramanujan deduces (F) from two false propositions

(G) $\quad \varphi = \varphi_1 - \varphi_2 = o\left(\frac{1}{y}\right)$ (false)

(H) $\quad \varphi_2 \sim \frac{1}{y}$ (false)

His actual statement is the wilder form of (G) in which $o(\frac{1}{y})$ is replaced by $O(1)$. In the case of (H) he may have reasoned by a false analogy between series and integrals for

(5.9) $$\log 2 \int_0^{\infty} 2^x e^{-2^x y}\, dx = \frac{1}{y}$$

The inference from (E) to (G) is invalid, as we show later. However, let us pursue Ramanujan's argument to its conclusion. If we now suppose, as is true

(5.10) $$\varphi_1(y) \sim \frac{1}{y}$$

then

(5.11) $$\int^{\infty} \log z \left(e^{-zy} + e^{-z^2 y} + e^{-z^3 y} + \cdots\right) d\,\pi(z) = \sum \Lambda(n)\, e^{-ny}$$

Neglecting the terms $e^{-z^2 y} + \ldots$ we have

(5.12) $$\int^{\infty} \log z \cdot e^{-zy}\, d\,\pi(z) \sim \frac{1}{y}$$

If we treat $\pi(z)$ as a differentiable function, then

(5.13) $$d\,\pi(z) = \frac{\chi(z)\, dz}{\log z}$$

[This seems crude, but the essential errors do not lie here.]

Now from

(5.14) $$\int^\infty \chi(z) e^{-zy}\, dz \sim \frac{1}{y} \quad \text{as} \quad y \to 0$$

he infers that $X(z) \to 1$. This inference would not be correct in any case, even by the use of the Hardy Littlewood Tauberian theorem. It would, however, follow (using the integral analogue of that theorem) th

(5.15) $$\int^z \chi(u)\, du \sim z$$

and the completion of the proof would then be easy.

Actually Ramanujan goes further. He makes use of the extra terms $e^{-z^2 y}$ ... and infers that

(5.16) $$\chi(z) = 1 - \tfrac{1}{2} z^{-\frac{1}{2}} - \tfrac{1}{3} z^{-\frac{2}{3}} \cdots \qquad \frac{d\,\pi(z)}{dz} = \frac{z - \frac{1}{2} z^{\frac{1}{2}} - \frac{1}{3} z^{\frac{1}{3}} \cdots}{z \log z}$$

whence by integration he obtains the Riemann-Ramanujan formula

(5.17) $$\pi(x) = \sum \frac{\mu(m)}{m}\, li\, x^{\frac{1}{m}}$$

It is not worth while, in view of previous mistakes, to examine this part of the argument in detail.

Can one say that apart from the necessary rigor this contains the essentials of the Hardy-Littlewood proof? No, because of Ramanujan's howler on the function $\varphi_2(y)$. His statement that

(5.18) $$\varphi_2(y) \sim \frac{1}{y}$$

is false as may be shown by directly considering its properties.

If it were true that

(5.19) $$\sum 2^m e^{-2^m y} \sim \frac{1}{y}$$

then it would follow, by Theorem 4.2 that

(5.20) $$\sum_{2^m \le x} 2^m \sim x$$

This is obviously false, since the function practically doubles its value when x passes through a value 2

It is instructive to examine $\varphi_2(y)$ more closely. Now

(5.21) $$e^{-2^m y} = \frac{1}{2\pi i}\int_C 2^{-ms} y^{-s}\, \Gamma(s)\, ds$$

where the integration is upwards on a straight line to the right of $\sigma = 1$. Hence

(5.22) $$2^m e^{-2^m y} = \frac{1}{2\pi i}\int_C 2^{m(1-s)} y^{-s}\, \Gamma(s)\, ds$$

and summing over m

(5.23) $$\varphi_2(y) = \frac{1}{2\pi i}\int_C y^{-s}\, \Gamma(s)\, \frac{2^{1-s}}{1-2^{1-s}}\, ds$$

We now move the contour of integration across the line $\sigma = 1$ and calculate the appropriate residues. For $s = 1$ we obtain

(5.24) $$\frac{1}{y} + \sum_{p=0}^{\infty} \frac{(-1)^p y^p}{p!}\, \frac{2^{1-p}}{1-2^{1+p}}$$

The other residues on $\sigma = 1$ are for $s = 1 + \frac{2k\pi i}{\log 2}$, and these yield the corrections to be added

$$+\sum \Gamma\left(\frac{1+2k\pi i}{\log 2}\right) y^{-1-\frac{2k\pi i}{\log 2}}$$

Hence $\varphi_2(y) = \frac{1}{y}$ + rapidly convergent series. When $y \to 0$ the series for $k \neq 0$ are bounded multiples of $\frac{1}{y}$ since the Γ function introduces factors which are less than $e^{-\frac{k\pi^2}{\log 2}}$ which is in turn less than e^{-1}
The terms in k are of the form

(5.25) $$\frac{1}{y}\left(A \cos\left\{\frac{2k\pi}{\log 2}\log y\right\} + B \sin\left\{\frac{2k\pi}{\log 2}\log y\right\}\right)$$

and hence

(5.26) $$\varphi_2(y) = \frac{1}{y} + \text{wobbly function of order } \frac{1}{y}.$$

But even without the use of the Mellin integral and the theory of contour integration it may be seen that Ramanujan's assertion is false. Let

(5.27) $$f(y) = \sum_{1}^{\infty} 2^m e^{-2^m y}$$

Then

(5.28) $$f(y) - 2f(2y) = 2e^{-2y}$$

Write, furthermore,

(5.29) $$g(y) = \frac{1}{y \log 2} + \sum_{0}^{\infty} \frac{(-1)^n y^n}{n!} \frac{2^{n+1}}{1-2^{n+1}}$$

Here also

(5.30) $$g(y) - 2g(2y) = 2e^{-2y}$$

Hence if

(5.31) $$h(y) = f(y) - g(y) \quad \text{where } h(y) \neq \text{constant}$$

then

(5.32) $$h(y) - 2h(2y) = 0$$

Put

(5.33) $$h(y) = \frac{1}{y} H(\log y)$$

and then

(5.34) $$H(\log y) = H(\log y + \log 2)$$

Hence H is periodic and does not tend to a limit as $y \rightarrow 0$ and consequently $h(y)$ is not asymptotic to a constant multiple of $\frac{1}{y}$. But $g(y) \sim \frac{1}{y \log 2}$ as $y \rightarrow 0$, and hence $f(y) = \frac{\varphi_2(y)}{\log 2}$ is not asymptotic to a constant multiple of $\frac{1}{y}$.

It is possible to use the same methods by which we showed that Ramanujan's assertion (H) on $\varphi_2(y)$ is false in order to prove that the corresponding assertion (5.10) on $\varphi_1(y)$ is true. From this we may derive a proof of the prime number theorem. We begin with the Mellin formula

(5.35) $$e^{-ny} = \frac{1}{2\pi i} \int_C y^{-s} n^{-s} \Gamma(s)\, ds$$

where the contour of integration is upwards on a straight line with $\sigma > 1$. From (5.35) we obtain

(5.36) $$\varphi_1(y) = \sum \Lambda(n) e^{-ny} = \frac{1}{2\pi i} \int_C \Gamma(s) y^{-s} \frac{\zeta'(s)}{\zeta(s)}\, ds$$

But using this representation it can be shown (Hardy and Littlewood, Acta Mathematica, vol. 41 (1917), pp. 119-196, that

(5.37) $$\varphi_1(y) \sim \frac{1}{y}$$

But this relation, taken with the Tauberian Theorem 4.2, enables us to conclude

(5.38) $$\psi(x) = \sum_{n \leq x} \Lambda(n) \sim x$$

which as we know is equivalent to the prime number theorem.

It may seem curious that although, as we have shown,

(5.39) $$\varphi_2(y) = \frac{\log 2}{2\pi i} \int \Gamma(s) y^{-s} \frac{2^{1-s}}{1-2^{1-s}}\, ds = \frac{1}{y} + \text{wobbly terms of order } \frac{1}{y}$$

nevertheless in

(5.40) $$\Phi_2(y) = \varphi_2(y) - \varphi_2(2y) + \varphi_2(3y) \cdots = \frac{2 \log 2}{e^{2y}-1} \text{ (from (A))}$$

the wobbles disappear. But this is understandable when we form the Mellin integral for $\Phi_2(y)$. This may be done by replacing y^{-s} in (5.39) by $y^{-s} - (2y)^{-s} + (3y)^{-s} \ldots$, and we obtain

(5.41) $$\Phi_2(y) = \frac{1}{2\pi i} \int \Gamma(s) y^{-s} \frac{2^{1-s}}{1-2^{1-s}} (1-2^{1-s}) \zeta(s)\, ds = \frac{1}{2\pi i} \int \Gamma(s) y^{-s} 2^{1-s} \zeta(s)\, ds$$

Here the dominant term comes from the pole $s = 1$, while the poles corresponding to the complex zeros of $1 - 2^{1-s}$ have dropped out completely. Hence

(5.42) $$\Phi_2(y) \sim \frac{1}{y}$$

Ramanujan inferred from

(5.43) $$\chi(y) - \chi(2y) + \chi(3y) \ldots \rightarrow \text{limit}$$

that

(5.44) $$\chi(y) \rightarrow \text{limit.}$$

As we have just seen, such inferences are invalid, even though there are Tauberian theorems of the Lambert type which bear some resemblance to such an inference. That the falsity of Ramanujan's assertion is due to the complex zeros of $1 - 2^{1-s}$ is well illustrated by the following example. It is possible that

(5.45) $$\chi(y) - \chi(2y) + \chi(3y) \ldots \equiv 0$$

with $\chi(y)$ a wobbly function of order $\frac{1}{y}$. For take $\chi(y) = y^{-1-ai}$. Then the left side of (5.45) becomes

(5.46) $$y^{-1-ai}(1 - 2^{-1-ai} + 3^{-1-ai} \ldots) = y^{-1-ai}\,\zeta(1 + ai)(1 - 2^{ai})$$

and this is identically zero whenever ai is one of the complex zeros of $1 - 2^{-s}$, say $a = \frac{2k\pi}{\log 2}$. Hence we may say that the failure of Ramanujan's attempt to prove $\varphi_1(y) \sim \frac{1}{y}$ is due to his ignoring the complex zeros of $1 - 2^{-s}$.

It is an interesting question to ask whether Ramanujan thought out this method entirely by himself, and in particular to inquire if he ever had seen the Riemann series approximating $\pi(x)$. From what available evidence there is, it seems highly probable that this work is entirely original and that Ramanujan had never seen the Riemann series. These questions depend upon what books there were in the library of the University of Madras dealing with the subject and whether Ramanujan actually saw any of them. There were five books, all probably at Madras, which would have been particularly valuable to Ramanujan: Whittaker's "Modern Analysis", Bromwich's "Infinite Series", Cayley's and Greenhill's "Elliptic Functions", and Mathews's "Theory of Numbers". It is quite clear that there were some of these books which Ramanujan had not seen. In particular he could not have seen Whittaker, because he did not know Cauchy's Theorem, and it seems extremely improbable that he could have seen Bromwich (since he knew nothing of the ordinary theories of divergent series).

On the other hand he must have read some book on elliptic functions. He never uses language which suggests that he regarded any of the standard theorems of elliptic functions as his own, but treats theta-functions, modular equations, and so forth, as common knowledge. And both his knowledge and his ignorance of the subject of Elliptic Functions fit in with a familiarity with Cayley and Greenhill. These two books scarcely consider the function-theoretic aspects of elliptic functions, but devote a great deal of space to the formal operations. Greenhill in fact adopts a somewhat eccentric approach which is clearly mirrored in Ramanujan.

Now if Ramanujan had seen Mathews's book, he would certainly have devoured it. But neither his knowledge nor ignorance of the Theory of Numbers are explicable if he had seen this book. He had, for example, no knowledge of the theory of quadratic forms, although Mathews devotes a great deal of space to it. In particular we must now ask whether or not he saw the chapter on the distribution of primes, which includes a copy of Riemann's paper. Here is given Riemann's formula

(5.47) $$\pi(x) = \sum \frac{\mu(m)}{m}\, li\, x^{\frac{1}{m}} + (\rho \text{ terms})$$

and the complex zeros of $\zeta(s)$ are clearly recognized. Yet we see that the failure of Ramanujan's method is due essentially to the ignoring of complex zeros.

It is quite inconceivable that a mathematician of Ramanujan's caliber could have seen Riemann's formula in this form and then could attempt a proof which completely ignores complex zeros.

It is perhaps worth while to give the broad outlines of Riemann's argument. From

(5.48) $$\Pi(x) = \sum_{p^m \leq x} \frac{1}{m}$$

he observes that

(5.49) $$\int_0^\infty \Pi(x)\, x^{-s-1}\, dx = \frac{1}{s}\int_0^\infty x^{-s}\, d\,\Pi(x) = \frac{1}{s}\sum \frac{1}{m\, p^{ms}} = \frac{\log \zeta(s)}{s}$$

This he inverts by using a Fourier double integral formula. Putting s = c + it it becomes

(5.50) $$\int_0^\infty \Pi(x)\, e^{-it\log x}\, dx = \frac{\log \zeta(s)}{s}$$

and an application of Fourier's theorem gives

(5.51) $$\Pi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\log \zeta(s)}{s}\, x^{s}\, ds$$

Thus we see that as $\psi(x)$ corresponds to $\frac{\zeta'(s)}{\zeta(s)}$ so $\Pi(x)$ corresponds to $\log \zeta(s)$. We can do the same thing in a more modern way using Mellin's inversion formulae

(5.52) $$\begin{aligned} f(s) &= \int_0^\infty x^{s-1} F(x)\, dx \\ F(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} f(s)\, ds \end{aligned}$$

Riemann now substitutes for $\zeta(s)$ its canonical product expression and evaluates the integral by term-by-term integration. The most serious gap in his argument lies just here, since the product expression for $\zeta(s)$ was only proved rigorously by Hadamard, and depends on his theory of integral functions.

Riemann's way would have seemed more natural to Ramanujan than the classical methods of proof of the prime number theorem and we are forced to the conclusion that Ramanujan would not have attempted his proof had he known of this method of Riemann's.

6. The Wiener-Ikehara proof of the prime number theorem

THEOREM 6.1. Let us suppose:

(1) $d\alpha(\xi) \geq 0$

(2) $f(s) = \int_0^\infty e^{-s\xi} d\alpha(\xi)$ is absolutely convergent for $\sigma > 1$.

(3) $f(s) - \frac{1}{s-1}$ is regular for $\sigma \geq 1$ and $|t| \leq T$.

Then we may conclude that $\alpha(\xi) \sim e^{\xi}$

We note first of all that there is <u>no</u> assumption as to the rate of growth of f(s) and the behavior of f(s) at infinity is in no manner specified.

Next, before proceeding to the proof, we note that the theorem may be formally translated so as to include the prime number theorem. Put $e^{\xi} = x$, $\alpha(\xi) = \beta(x)$ Then

(6.1) $$f(s) = \int_1^\infty x^{-s}\, d\beta(x)$$

and the conclusion is that $\beta(x) \sim x$. In particular, if $\beta(x) = \psi(x)$, we have

(6.2) $$f(s) = \int_1^\infty x^{-s}\, d\,\psi(x)$$

which is the series

(6.3) $$f(s) = \sum \frac{\log p}{p^{ms}} = -\frac{\zeta'(s)}{\zeta(s)}$$

and the condition that $f(s) - \frac{1}{s-1}$ be regular for $\sigma \geq 1$ is that

(6.4) $$\zeta(1 + it) \neq 0$$

which is an essential to all known proofs of the prime number theorem.

To prove the theorem we observe that if we put

(6.5) $$g(\xi) = e^{-\xi}\alpha(\xi) - 1$$

the conclusion is equivalent to

(6.6) $$g(\xi) \to 0 \text{ as } \xi \to \infty$$

We make use of the Fejer kernel

(6.7) $$K_\lambda(y) = \frac{\sin^2 \lambda y}{\lambda y^2} \qquad (\lambda > 0)$$

and write

(6.8) $$g_\lambda(\xi) = \int_0^\infty K_\lambda(\xi - \eta)\, g(\eta)\, d\eta$$

It is true that for a general g(x)

(6.9) $$g_\lambda(\xi) \to C\, g(\xi) \qquad \text{as } \lambda \to \infty$$

and we may regard $g_\lambda(\xi)$ as a kind of average of $g(\xi)$.

We prove the theorem by showing (a) that $g_\lambda(\xi) \to 0$ when $\xi \to \infty$ for every λ, and (b) that from this $g(\xi) \to 0$ follows. The second step is the "Tauberian" part of the proof.

We take (b) first. Assume it proved that $g_\lambda(\xi) \to 0$ for every λ. Suppose that

$$\overline{\lim_{\xi\to\infty}}\, g(\xi) = L \quad \text{and} \quad \underline{\lim_{\xi\to\infty}}\, g(\xi) = \ell.$$

We must show that $L \le 0$ and $\ell \ge 0$. Suppose first, on the contrary, that $L > 0$. Then $\alpha(\eta) \ge \alpha(\xi)$ for $\eta \ge \xi$ whence

(6.10) $$e^\eta (g(\eta)+1) \ge e^\xi (g(\xi)+1)$$

and

(6.11) $$g(\eta) \ge e^{\xi-\eta} g(\xi) + e^{\xi-\eta} - 1.$$

Suppose $g(\xi) \ge \frac12 L$ for a sequence of values x of ξ tending to ∞. Then there exists a δ such that

(6.12) $$g(\eta) > \tfrac14 L$$

for η in a series of ξ intervals $(x, x+\delta)$ with fixed δ and x tending to infinity. On the other hand

(6.13) $$g(\eta) = \frac{\alpha(\eta)}{e^\eta} - 1 > -B$$

since $d\alpha(\xi) \ge 0$. Taking $\xi = x + \frac12\delta$

(6.14) $$g_\lambda(\xi) = \int_0^\infty \frac{\sin^2\lambda(\xi-\eta)}{\lambda(\xi-\eta)^2}\, g(\eta)\, d\eta > \frac14 L \int_{\xi-\frac12\delta}^{\xi+\frac12\delta} \frac{\sin^2\lambda(\xi-\eta)}{\lambda(\xi-\eta)^2}\, d\eta - B\Big[\int_0^{\xi-\frac12\delta} + \int_{\xi+\frac12\delta}^{\infty}$$

But for fixed δ

(6.15) $$\int_{\xi-\frac12\delta}^{\xi+\frac12\delta} \frac{\sin^2\lambda(\xi-\eta)}{\lambda(\xi-\eta)^2}\, d\eta \to \pi \quad \text{and} \quad \int_0^{\xi-\frac12\delta},\ \int_{\xi+\frac12\delta}^{\infty} \to 0$$

when $\lambda \to \infty$. Hence if λ is large enough

(6.16) $$g_\lambda(\xi) > \frac{\pi}{8} L$$

for ξ surpassing all limit and this contradicts $g_\lambda(\xi) \to 0$ for λ fixed as $\xi \to \infty$. Similarly we may eliminate the assumption $\ell < 0$.

It remains to prove that (1) $g_\lambda(\xi)$ exists and (2) $g_\lambda(\xi) \to 0$ for any fixed λ. Write $s = 1 + \delta + it$, $\delta > 0$ and also

(6.17) $$h_\delta(t) = \int_0^\infty g(\xi)\, e^{-\delta\xi - it\xi}\, d\xi$$

This is the Fourier transform of $\sqrt{2\pi}\, g(\xi) e^{-\delta\xi}$ for $\xi > 0$ and 0 for $\xi < 0$.

Then

(6.18) $$\lim_{\delta\to0} h_\delta(t) = h(t) = \int_0^\infty (\alpha(\xi) - e^\xi)\, e^{-s\xi}\, d\xi = \frac{\alpha(0)}{s} + \frac1s \int_0^\infty e^{-s\xi}\, d\alpha(\xi) - \frac{1}{s-1}$$

Moreover, as $\delta \to 0$, $h_\delta(t) \to h(t)$ uniformly in $|t| \le T$. On the other hand

(6.19) $$K_\lambda(\xi) = \frac{\sin^2\lambda\xi}{\lambda\xi^2} = 2\int_{-2\lambda}^{2\lambda} \Big(1 - \frac{|t|}{2\lambda}\Big) e^{it\xi}\, dt$$

which is the Fourier transform of the function

$$(6.20) \qquad \begin{cases} 2\sqrt{2\pi}\left(1 - \frac{|t|}{2\lambda}\right) & \text{for } |t| < 2\lambda \\ 0 & \text{for } |t| \geq 2\lambda \end{cases}$$

We now appeal to Parseval's theorem which states that if F and G are the Fourier transforms of the (real) functions f and g, then

$$(6.21) \qquad \int_{-\infty}^{+\infty} fg\,dx = \int_{-\infty}^{+\infty} FG\,dx$$

provided that f and g belong to L^2. The Fourier transform is of course given by

$$(6.22) \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(t) e^{-xit}\,dt$$

If we replace x by x-y in this equation we have

$$(6.23) \qquad f(x-y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(t) e^{yit} e^{-xit}\,dt$$

whence f(x-y) is the Fourier transform of $F(x)e^{yix}$. We apply Parseval's Theorem to $f = K_\lambda(x-y)$ whose Fourier transform by (6.19), (6.20) and (6.23) is

$$(6.24) \qquad \begin{cases} \text{const.}\left(1 - \frac{|x|}{2\lambda}\right)e^{yix} & \text{for } |x| < 2\lambda \\ 0 & \text{for } |x| \geq 2\lambda \end{cases}$$

and to

$$g = \begin{cases} g(x)e^{-\delta x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

whose Fourier transform by (6.17) is $h_\delta(x)$. Substituting in (6.21) we have

$$(6.25) \qquad \int_0^\infty K_\lambda(x-y)g(x)e^{-\delta x}\,dx = \text{const} \int_{-2\lambda}^{+2\lambda} \left(1 - \frac{|x|}{2\lambda}\right) e^{yix} h_\delta(x)\,dx$$

Since $K_\lambda(x-y) \geq 0$, and is integrable and $g(x)e^{-\delta x}$ is bounded below, the integral

$$(6.26) \qquad \int_0^\infty K_\lambda(x-y)g(x)\,dx$$

is either (1) absolutely convergent or (2) diverges to $+\infty$. The second hypothesis is impossible because the right-hand side of (6.25) tends to a finite limit. Hence

$$(6.27) \qquad \begin{aligned} g_\lambda(\xi) &= \int_0^\infty K_\lambda(x-y)g(x)\,dx \\ &= \text{const} \int_{-\infty}^{+\infty} f(t)e^{yit}\,dt \end{aligned}$$

where the integrals are absolutely convergent. But the Riemann-Lebesgue theorem states that the second integral must under these circumstances tend to zero as $y \to \infty$. Hence $g_\lambda(y) \to 0$ as $y \to \infty$ for λ fixed and the proof of our theorem

$$(6.28) \qquad g(\xi) \to 0 \text{ as } \xi \to \infty$$

is now complete.

We can prove (6.25) without any appeal to Parseval's theorem. In fact we are given that

$$(6.29) \qquad \int_0^\infty e^{-(1+\delta)x}\,d\alpha(x) < \infty \quad \text{or} \quad \int_0^{x'} e^{-(1+\delta)x}\,d\alpha(x) < A$$

A fortiori $e^{-(1+\delta)x'}(\alpha(x') - \alpha(0)) < A$, $\alpha(x) = O(e^{(1+\delta)x})$, $g(x) = O(e^{\delta x})$ for every $\delta > 0$. Now

$$(6.30) \qquad \begin{aligned} \int_{-2\lambda}^{+2\lambda}\left(1 - \frac{|x|}{2\lambda}\right)e^{yix} \int_0^\infty g(\xi)e^{-\delta\xi - ix\xi}\,d\xi &= \int_0^\infty g(\xi)e^{-\delta\xi}\,d\xi \int_{-2\lambda}^{2\lambda}\left(1 - \frac{|x|}{2\lambda}\right)e^{(y-\xi)ix}\,dx \\ &= \text{const} \int_0^\infty g(\xi)e^{-\delta\xi} K_\lambda(y - \xi)\,d\xi \end{aligned}$$

(by "absolute and uniform" convergence).

Before leaving the subject of the distribution of primes, it is well to consider Ramanujan's other assertions on this subject. His assertion (1) that $\pi(x) < \text{li}\,x$ and his assertion (6) that

$\pi_{4n+3}(x) - \pi_{4n+1}(x) \longrightarrow \infty$ were shown to be false by Littlewood (Acta Mathematica, Vol. 41). It was supposed by many mathematicians that

$$\pi(x) = \mathrm{li}\,x - \tfrac{1}{2}\mathrm{li}\,x^{\frac{1}{2}} + \text{wobbly terms of order } \frac{x^{\frac{1}{2}}}{\log x} \tag{6.31}$$

and if these wobbly terms were either $o(\frac{x^{\frac{1}{2}}}{\log x})$ or $O(\frac{x^{\frac{1}{2}}}{\log x})$ where the constant in the O relation is not too large, then Ramanujan's assertion would be true. But Littlewood showed that these wobbly terms are no smaller than $K\frac{x^{\frac{1}{2}}}{\log x}\log\log\log x$ for at least some sequence of x's tending to infinity. Littlewood also gave a similar result disproving Ramanujan's assertion (6). These assertions are however true in (C, 1) average providing that the Riemann Hypothesis (or its generalization for the other Dirichlet series used in the theory) is true. These modified assertions stand or fall with the Riemann Hypothesis and its generalizations.

7. The right-angled triangle problem and allied problems

Let us turn to Ramanujan's eighth assertion: that if N(x) is the number of numbers less than x of the form

$$2^u 3^v \tag{7.1}$$

then approximately

$$N(x) = \frac{1}{2}\,\frac{\log 2x \log 3x}{\log 2 \log 3}. \tag{7.2}$$

This is an elegantly disguised form of the assertion

$$N(x) = \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} + \frac{1}{2} \tag{7.3}$$

where

$$\eta = \log x, \qquad \omega = \log 2, \qquad \omega' = \log 3. \tag{7.4}$$

The first three terms are correct, but if the use of any constant term is justified it is not $\frac{1}{2}$ but

$$\frac{\omega^2 + \omega'^2 + 3\omega\omega'}{12\,\omega\omega'} \tag{7.5}$$

The assertions (7.2) or (7.3) are particular cases of the problem of lattice points within a right-angled triangle. Let $M(\eta) = N(x)$ be the number of positive pairs of integers (u, v) satisfying

$$u\,\omega + v\,\omega' \le \eta \tag{7.6}$$

It has been proved (Hardy-Littlewood P.L.M.S. vol. 20) that

$$M(\eta) = \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} + o(\eta) \tag{7.7}$$

for all irrational $\frac{\omega}{\omega'}$. The error term may be improved to $O(\eta^{\alpha})$ $0 < \alpha < 1$ for all algebraic irrational $\frac{\omega}{\omega'}$ and improved still further to $O(\log \eta)$ when the quotients in the continued fraction for ω/ω' are bounded, hence in particular for quadratic irrationalities. It is simple to show that if $\theta = \omega/\omega'$ is rational, then the wobbles in the value of $M(\eta)$ are $O(\eta)$.

We may prove

$$M(\eta) = \frac{\eta^2}{2\omega\omega'} + O(\eta) \qquad \text{for all } \theta \tag{7.8}$$

very easily. Geometrically this means that the number of lattice points in a triangle is the area of the triangle with an error whose order is not greater than that of the circumference. For

$$\begin{aligned} M(\eta) &= \sum_{\substack{u,v>0 \\ u\omega+v\omega'\le\eta}} 1 = \sum_{u\omega\le\eta}\left[\frac{\eta-u\omega}{\omega'}\right] = \sum_{u\omega\le\eta}\frac{\eta-u\omega}{\omega'} + O(\eta) \\ &= \frac{\eta}{\omega'}\left[\frac{\eta}{\omega}\right] - \frac{\omega}{2\omega'}\left[\frac{\eta}{\omega}\right]\left(\left[\frac{\eta}{\omega}\right]+1\right) + O(\eta) = \frac{\eta^2}{2\omega'\omega} + O(\eta) \end{aligned} \tag{7.9}$$

But nothing further than this is trivial.

Let us investigate more carefully Ramanujan's special case (7.4). Here

$$\theta = \frac{\log 3}{\log 2} \tag{7.10}$$

is irrational and also transcendental, for

(7.11) $$2^{\theta} = 3$$

and if θ were rational we should have a power of 3 equal to a power of 2. It must also be transcendental since 2^{α}, where α is an irrational algebraic number, must be transcendental and hence in particular is not the number 3. Suppose that

(7.12) $$v = 2^p 3^q = e^{p\omega + q\omega'} = e^{\lambda}$$

Then

(7.13) $$\sum_{\lambda} e^{-\lambda s} = \sum_{v} v^{-s} = \sum_{p,q} e^{-s(p\omega + q\omega')} = \frac{1}{(1-e^{-s\omega})(1-e^{-s\omega'})}$$

From this and the Perron formula

(7.14) $$\sum_{\substack{\lambda \le \eta \\ [\text{i.e. } v \le x]}} 1 = \frac{1}{2\pi i}\int \frac{x^s}{s}\cdot\frac{1}{(1-2^{-s})(1-3^{-s})}\,ds$$

We may calculate this formally using Cauchy's formula. As there is a triple pole at the origin, the calculus of residues yields as approximation

(7.15) $$M(\eta) = \alpha\eta^2 + \beta\eta + \gamma$$

and computing α, β, γ

(7.16) $$M(\eta) = \frac{\eta^2}{2\omega\omega'} + \eta\left(\frac{1}{2\omega} + \frac{1}{2\omega'}\right) + \frac{\omega^2 + \omega'^2 + 3\omega\omega'}{12\omega\omega'} + S$$

where S consists of terms derived from the complex zeros of $1 - 2^{-s}$ and $1 - 3^{-s}$. Explicitly

(7.17) $$S = \frac{1}{2\pi}\sum_{k=1}^{\infty}{}^{*} \frac{\cos\left(\frac{2k\pi\eta}{\omega} + \frac{k\pi\omega'}{\omega}\right)}{k\sin\frac{k\pi\omega'}{\omega}}$$

where the * means to add similar terms obtained by interchanging ω and ω'. This series is convergent only in some very Pickwickian sense. It is not too hopelessly divergent when θ is algebraic.

There is another interesting arithmetic question which depends upon the number $\theta = \frac{\log 3}{\log 2}$ of (7.10). S. S. Pillai, one of the best of the Indian mathematicians, has proved that

(7.18) $$|2^x - 3^y| < 2^{x(1-\delta)}$$

for any given δ and $x > X(\delta)$. Stated otherwise

(7.19) $$|2^x 3^{-y} - 1| > K 2^{-\delta x}$$

for every δ and some $K = K(\delta)$. This becomes

(7.20) $$|e^{x\omega - y\omega'} - 1| > Ke^{-\delta x} \quad\text{or}\quad |x\omega - y\omega'| > Ke^{-\delta x}$$

whence

(7.21) $$|x - y\theta| > Ke^{-\delta x}$$

Liouville proved for algebraic numbers α that

(7.22) $$|x - y\alpha| > \frac{K}{x^{\lambda}}$$

where λ depends on α. This gives a limit to the rate of growth of a_n, the n-th quotient to θ, of the form

(7.23) $$a_{n+1} = O(e^{\epsilon} a_1 a_2 \cdots a_n)$$

but this still permits an enormously rapid increase.

The fact that

(7.24) $$2^x - 3^y = k$$

has only a finite number of solutions (x, y) for every k is an immediate corollary of the Thue-Siegel Theorem. For if we reduce x and y modulo 3 in (7.24) we obtain nine equations of the type

(7.25) $$au^3 - bv^3 = k \qquad u = 2^{\bar{x}},\ v = 3^{\bar{y}}$$

and each of these has only a finite number of solutions. From this we deduce at once

(7.26) $$|2^x - 3^y| \to \infty \quad \text{as } x \to \infty$$

but Pillai's result (7.18) is much stronger than this.

Formally stated, his theorem is

THEOREM 7.1. Suppose that m, n, a, b are given integers and that $\frac{\log m}{\log n}$ is irrational. Then for any $\delta > 0$

$$|am^x - bn^y| > m^{x(1-\delta)} \tag{7.27}$$

provided that $x > X(\delta)$.

Proof:

Lemma 1. If ξ is any algebraic number of degree $r \geq 3$, there exists a $K = K(\xi)$ such that

$$\left|\xi - \frac{p}{q}\right| > \frac{K}{q^{2\sqrt{r}}} \tag{7.28}$$

This is the Thue-Siegel Theorem.

Lemma 2. If a, b, and r are given, then there exists a $K(a, b, r)$ such that

$$|au^r - bv^r| > Ku^{r-2\sqrt{r}} \qquad (r \geq 3). \tag{7.29}$$

To prove this we assume $au^r - bv^r > 0$. [For < 0 simply interchange a, b; u, v.] Now if $bv^r < \frac{1}{2} au^r$, (7.29) is certainly satisfied for $K = \frac{1}{2}$. Hence we may assume

$$\tfrac{1}{2} au^r < bv^r < au^r \tag{7.30}$$

$$au^r - bv^r = P.$$

Here if we write $\alpha = \sqrt[r]{\frac{a}{b}}$, $w = \alpha u$ we have

$$w^r - v^r = \frac{P}{b} \tag{7.31}$$

whence

$$w - v < \frac{P}{brv^{r-1}} \quad \text{or} \quad \alpha u - v < \frac{P}{brv^{r-1}} \tag{7.32}$$

or

$$\left|\alpha - \frac{v}{u}\right| < \frac{P}{bruv^{r-1}} \tag{7.33}$$

but from Lemma 1, since α is algebraic of degree r

$$\left|\alpha - \frac{v}{u}\right| > \frac{K(r)}{u^{2\sqrt{r}}} \tag{7.34}$$

whence from (7.33)

$$\frac{P}{buv^{r-1}} > \frac{K(r)}{u^{2\sqrt{r}}} \tag{7.35}$$

or

$$P > \frac{K(r)uv^{r-1}}{u^{2\sqrt{r}}} > K(r)u^{r-2\sqrt{r}} \tag{7.36}$$

If in

$$N = am^x - bn^y \tag{7.37}$$

we write

$$\begin{aligned} x &= rs + h \qquad & 0 \leq h, \ell < r \\ y &= rt + \ell \qquad & u = m^s, \quad v = n^t \end{aligned} \tag{7.38}$$

we have

$$N = am^h u^r - bn^\ell v^r \tag{7.39}$$

and by Lemma 2

$$|N| > Ku^{r-2\sqrt{r}} \tag{7.40}$$

where K depends on a, b, m, n, r, h, and ℓ. For a fixed r there are only a finite number of h and ℓ. If we choose the least K we have the relation (7.40) where K depends on a, b, m, n, and r. Substituting in (7.37),

$$|am^x - bn^y| > u^{r-3\sqrt{r}} \qquad \text{for } x > X(r) \tag{7.41}$$

where $u^r = m^{rs} = m^{x-h} > K(r)m^x$. Hence

$$|am^x - bn^y| > K(r)m^{x(1-\frac{3}{\sqrt{r}})} \tag{7.42}$$

Here we need only choose r to make $\left|\frac{3}{\sqrt{r}}\right| < \delta$ and the theorem is proved.

Ramanujan's formulae (9) contain statements greatly differing in depth. Let ν be a number for which $\mu(\nu) = -1$ and λ be a number for which $\mu(\lambda) = +1$. Then the set of all ν's and λ's consists of all square-free numbers q. Let us first prove (b) which is elementary. It is well known that

(7.43) $$\sum \frac{1}{q^2} = \frac{\zeta(2)}{\zeta(4)} = \sum \frac{1}{\lambda^2} + \sum \frac{1}{\nu^2} \quad \text{and} \quad \sum \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)}$$

But $\sum \frac{\mu(n)}{n^2} = \sum \frac{1}{\lambda^2} - \sum \frac{1}{\nu^2}$. Hence $\sum \frac{1}{\nu^2} = \frac{1}{2}\left(\frac{\zeta(2)}{\zeta(4)} - \frac{1}{\zeta(2)}\right)$. But $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$

Hence $\sum \frac{1}{\nu^2} = \frac{1}{2}\left(\frac{\frac{\pi^2}{6}}{\frac{\pi^4}{90}} - \frac{6}{\pi^2}\right) = \frac{9}{2\pi^2}$. On the other hand (a) involves a problem of great depth.

Let $Q(x) = \sum_{q \le x} 1$, $N(x) = \sum_{\nu \le x} 1$, $L(x) = \sum_{\lambda \le x} 1$. Hence $Q(x) = N(x) + L(x)$. Moreover $M(x) = \sum_{n \le x} \mu(n) = \sum_{\lambda \le x} 1 - \sum_{\nu \le x} 1 = L(x) - N(x)$. Hence to prove $N(x) = \frac{1}{2}(Q(x) - M(x)) \sim \frac{3x}{\pi^2}$ it is suffixient to prove

(7.44) $$Q(x) = \frac{6x}{\pi^2} + o(x) \text{ and}$$
$$M(x) = o(x).$$

It is elementary to prove that

(7.45) $$Q(x) = \frac{6x}{\pi^2} + O(\sqrt{x})$$

[Landau, "Primzahlen", vol. 2, p. 581] but the proof of

(7.46) $$M(x) = o(x)$$

is equivalent to the prime number theorem

$$\pi(x) \sim \frac{x}{\log x}$$

and is a problem of much greater depth than the result for Q(x).

If we interpret Ramanujan's tenth statement

(7.47) $$d(1) + d(2) + \dots + d(n) = n \log n + (2\gamma - 1)n + \tfrac{1}{2}d(n)$$

to mean that

(7.48) $$d(1) + d(2) + \dots + d(n) = n \log n + (2\gamma - 1)n + O(d(n))$$

it is certainly false for

(7.49) $$d(n) = O(n^{\varepsilon}) \text{ for any } \varepsilon > 0$$

and the error term in this formula is greater than this. Dirichlet has given an ingenious proof showing that in $d(1) + \dots + d(n) = n \log n + (2 \quad - 1)n + O(?)$ the error is $O(\sqrt{n})$. Now

(7.50) $$\sum_{m \le n} d(m) = \sum_{1 \le xy \le n} 1 = \sum_{\substack{1 \le xy \le n \\ 1 \le x \le \sqrt{n}}} 1 + \sum_{\substack{1 \le xy \le n \\ 1 \le y \le \sqrt{n}}} 1 - \sum_{\substack{1 \le x \le \sqrt{n} \\ 1 \le y \le \sqrt{n}}} 1 = \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_{12} = 2\sum\nolimits_1 - \sum\nolimits_{12}$$

by reasons of symmetry. Moreover

(7.51) $$\sum\nolimits_1 = \sum_{x \le \sqrt{n}} \left[\frac{n}{x}\right] = n \sum_{x \le \sqrt{n}} \frac{1}{x} + O(\sqrt{n}) = n\left(\tfrac{1}{2}\log n + \gamma + O\left(\frac{1}{\sqrt{n}}\right)\right) + O(\sqrt{n}).$$

Hence $2\sum_1 = n \log n + 2\gamma n$ and $\sum_{12} = [\sqrt{n}]^2 = (\sqrt{n} + O(1))^2 + O(\sqrt{n}) = n + O(\sqrt{n})$. Finally

(7.52) $$\sum_{m \le n} d(m) = 2\sum\nolimits_1 - \sum\nolimits_{12} = n \log n + n(2\gamma - 1) + O(\sqrt{n})$$

Voroni has obtained $O(n^{\frac{1}{3}})$ by a refinement of Dirichlet's method. Suppose that θ is the best possible order of error, that is the error is $O(n^{\theta + \varepsilon})$ for every $\varepsilon > 0$. Dirichlet's argument gives $\theta \le \frac{1}{2}$, Voronoi $\theta \le \frac{1}{3}$, Landau and Hardy $\theta \ge \frac{1}{4}$, van der Corput and others $\theta \le \frac{37}{112}, \frac{33}{100}, \frac{27}{82}, \frac{15}{46}$ (?).

A problem of equal depth is Gauss's circle problem. Let r(n) denote the number of solutions of

(7.53) $$u^2 + v^2 = n$$

Then $r(1) + \dots + r(n) = \pi n + O(?)$. Gauss showed $O(\sqrt{n})$, as might be expected. Every method so far applied to one of these problems has been applicable to the other and the same restrictions have been found on the order of the error term.

Ramanujan's eleventh assertion is

(7.54) $\nu = u^2 + v^2 \qquad B(x) = \sum_{\nu \le x} 1 = C \int_1^x \frac{dy}{\sqrt{\log y}} + R(x),$

where $R(x)$ is small in comparison with the integral, i.e.

(7.55) $$R(x) = o\left(\frac{x}{\sqrt{\log x}}\right)$$

if this is all that is meant, the integral is no improvement on

(7.56) $$B(x) = \frac{Cx}{\sqrt{\log x}} + o\left(\frac{x}{\sqrt{\log x}}\right)$$

Taken literally this statement is true, but his use of the integral should certainly imply the stronger relation

(7.57) $$B(x) = \frac{Cx}{\sqrt{\log x}} + \frac{C_1 x}{(\log x)^{3/2}} + \dots + O\left(\frac{x}{(\log x)^{r}}\right)$$

and this is false, as has been shown by G. K. Stanley in the third volume of the Journal of the London Mathematical Society.

8. The theory of partitions

A partition of n is a division of n into any number of positive integral parts: thus

(8.1) $$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$$

has 7 partitions. Order is irrelevant, so that we may think of the parts, if we please, as arranged in decreasing order. We denote the number of partitions of n by $p(n)$. Thus $p(1) = 1$ and $p(5) = 7$. A partition may be represented by an array of dots, or "nodes", such as

(8.2)

• • • • • • •
• • • •
• • •
• • •
•

a row corresponding to a part: thus (8.2) represents the partition $7 + 4 + 3 + 3 + 1$ of 18. A graph such as (8.2) may also be read vertically, here as $5 + 4 + 4 + 2 + 1 + 1 + 1$, and two partitions of n so related are called conjugate.

There are many theorems about partitions which may be proved by strictly elementary reasoning based upon this graphical representation. Thus a graph with m rows represents, when read horizontally, a partition into m parts, while read vertically it represents a partition into parts the largest of which is m. It follows that the number of partitions of n into m parts (or at most m parts) is also the number of partitions into parts of which the largest is m (or into parts which do not exceed m). But when we attack more difficult problems, we find very quickly that we require algebraic or analytic weapons.

In the theory of primes the multiplicative property of Dirichlet series

(8.3) $$m^{-s} \cdot n^{-s} = (mn)^{-s}$$

makes them the appropriate analytic tool. But in additive problems of arithmetic, the relation

(8.4) $$a^m \cdot a^n = a^{m+n}$$

makes power series the natural weapon.

The algebraic theory was founded by Euler, and rests on a generating power series

(8.5) $$F(q) = \sum f(n) q^n$$

said to enumerate $f(n)$.

It is easy to find the generating function of $p(n)$. This is the function

(8.6) $$F(q) = \frac{1}{(1-q)(1-q^2)(1-q^3)\dots}$$

(a function fundamental in the theory of elliptic functions). In fact, expanding each factor of $F(q)$, we have

(8.7) $$F(q) = (1+q+q^2+q^3+\dots)\cdot(1+q^2+q^4+q^6+\dots)\cdot(1+q^3+q^6+q^9+\dots)\ \dots\dots$$

and a moment's consideration shows that every partition of n contributes just 1 to the coefficient of q^n. Hence

(8.8) $$F(q) = \sum p(n)q^n.$$

It is equally easy to find the generating functions which enumerate partitions of n into parts restricted in various manners. Thus

(8.9) $$\frac{1}{(1-q)(1-q^3)(1-q^5)\dots}$$

enumerates partitions into <u>odd</u> parts;

(8.10) $$(1+q)(1+q^2)(1+q^3)\dots$$

partitions into unequal parts; and

(8.11) $$(1+q)(1+q^3)(1+q^5)\dots$$

partitions into parts both <u>odd and unequal</u>.

Similarly (these are examples which will be used later)

(8.12) $$\frac{1}{(1-q)(1-q^2)\dots(1-q^m)}$$

enumerates partitions into parts not exceeding m, or (what we have seen to be equivalent) into at most m parts;

(8.13) $$\frac{q^N}{(1-q^2)(1-q^4)\dots(1-q^{2m})}$$

enumerates the partitions of n-N into even parts; and

(8.14) $$\frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)\dots}$$

where the indices of q are the numbers 5m+1 and 5m+4, enumerates the partitions of n into parts each of which is of one of these forms.

Ramanujan was the first mathematician (and, it seems, up to now the only one) to discover any properly arithmetical property of the function p(n). His theorems were discovered, in the first instance, experimentally, i.e. by observation. MacMahon had calculated, for other purposes to which reference will be made later, a table of p(n) for the first 200 values of n, and Ramanujan observed that the table indicated certain simple "congruence properties" of p(n).

In particular, the number of partitions of numbers 5m+4, 7m+5, and 11m+6 are divisible by 5, 7, and 11 respectively: i.e.

(8.15) (a) $p(5m+4) \equiv 0 \pmod 5$ (b) $p(7m+5) \equiv 0 \pmod 7$ (c) $p(11m+6) \equiv 0 \pmod{11}$

Thus 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1 has 5 partitions.

Ramanujan found proofs of (a) and (b) so short that they will be given here; but they depend upon classical identities from the theory of elliptic functions. We start from a famous formula

(8.16) $$\prod_{k=1}^{\infty}(1-q^{2k})(1-q^{2k-1}z^2)(1-q^{2k-1}z^{-2}) = \sum_{-\infty}^{+\infty}(-1)^k z^{2k} q^{k^2}$$

of Gauss and Jacobi.

Suppose that

(8.17) $$\varphi(z) = \prod_{k=1}^{n}(1-q^{2k-1}z^2)(1-q^{2k-1}z^{-2})$$

Then we can verify at once that

(8.18) $$\varphi(qz)(qz^2+q^{2n}) = \varphi(z)(1+q^{2n+1}z).$$

On the other hand

(8.19) $$\varphi(z) = a_0 + a_1(z^2+z^{-2}) + a_2(z^4+z^{-4}) + \dots + a_n(z^{2n}+z^{-2n})$$

where the a's are polynomials in q and

(8.20) $$a_n = q^{1+3+\dots+2n-1} = q^{n^2}.$$

Substituting from (8.19) into (8.18) and equating the coefficients of z^{-2k+2}, we obtain

(8.21)
$$a_k = a_{k-1}\,\frac{q^{2k-1}(1-q^{2n-2k+2})}{1-q^{2n+2k}}$$

From (8.20) and (8.21) we can calculate a_k ; and we find

(8.22)
$$a_k = \frac{q^{k^2}}{(1-q^2)(1-q^4)\ldots(1-q^{2n})}\prod_{2n-2k+2}^{2n}(1-q^{2\nu})\prod_{2n+2k+2}^{4n}(1-q^{2\nu})$$

which tends to

(8.23)
$$\frac{q^{k^2}}{(1-q^2)(1-q^4)\ldots}$$

when $n \rightarrow \infty$. This gives (8.16) though a little care is necessary in justifying the passage to the limit. See Tannery and Molk "Fonctions 'elliptiques", T.2, pp. 10-12.

If in (8.16) we replace q, z by $q^{\frac{3}{2}}$, $q^{\frac{1}{4}}$ respectively, we obtain

(8.24)
$$\prod(1-q^{3k})(1-q^{3k-1})(1-q^{3k-2}) = \sum_{-\infty}^{+\infty}(-1)^k q^{\frac{3k^2+k}{2}} \qquad \text{or}$$

(8.25)
$$(1-q)(1-q^2)(1-q^3)\ \ldots = 1 - q - q^2 + q^5 + q^7 - \ldots$$

the indices on the right-hand side being $\frac{3}{2}k^2 \pm \frac{1}{2}k$. This formula, of which a good many proofs are known, was found first by Euler. There is a particularly simple combinatorial proof due to Franklin which will be given later.

Ramanujan used (8.25) and another formula of Jacobi, viz.

(8.26)
$$\{(1-q)(1-q^2)(1-q^3)\ \ldots\}^3 = 1 - 3q + 5q^3 - 7q^6 \ldots$$

where the exponents are the triangular numbers $\frac{1}{2}k(k+1)$. To prove this we replace q, z in (8.16) by $q^{\frac{1}{2}}$, $q^{\frac{1}{4}+\varepsilon}$ and make $\varepsilon \rightarrow 0$. One factor on the left is

(8.27)
$$1 - q^{\frac{1}{2}-\frac{1}{2}-2\varepsilon} \sim 2\varepsilon \log q$$

and the product behaves like

(8.28)
$$2\varepsilon \log q \prod_1^\infty(1-q^k)^2\prod_2^\infty(1-q^{k-1}) = 2\varepsilon \log q\left\{\prod_1^\infty(1-q^k)\right\}^3.$$

The series on the right becomes

(8.29)
$$\sum_{-\infty}^{+\infty}(-1)^k q^{\frac{1}{2}k^2}q^{\frac{1}{2}k+2k\varepsilon} = \sum_{-\infty}^{+\infty}(-1)^k q^{\frac{1}{2}k(k+1)}\ (1+2k\varepsilon \log q \ldots)$$

The term independent of ε vanishes, the terms for which k = r and k = -r-1 canceling, and the term in ε is

(8.30)
$$2\varepsilon \log q \sum_0^\infty(-1)^k(2k+1)q^{\frac{1}{2}k(k+1)}$$

which proves Jacobi's formula (8.26).

Ramanujan now argues as follows. We have

(8.31)
$$q\{(1-q)(1-q^2)\ldots\}^4 = q(1-q)(1-q^2)\ldots\{(1-q)(1-q^2)\ldots\}^3 = q(1-q-q^2+q^5+q^7\ldots)(1-3q+5q^3-7q^6\ldots)$$

by (8.25) and (8.26). We write this as

(8.32)
$$q\{(1-q)(1-q^2)\ldots\}^4 = \sum_\mu\sum_\nu(-1)^{\mu+\nu}(2\nu+1)q^{1+\frac{1}{2}\mu(3\mu+1)+\frac{1}{2}\nu(\nu+1)}$$

both μ and ν running from $-\infty$ to ∞ and we consider in what circumstances the index of q is divisible by 5. This demands that

(8.33)
$$2(\mu+1)^2 + (2\nu+1)^2 = 8\left\{1+\tfrac{1}{2}\mu(3\mu+1) + \tfrac{1}{2}\nu(\nu+1)\right\} - 10\mu^2 - 5$$

shall also be a multiple of 5. Now

(8.34)
$$(2\nu+1)^2 \equiv 0,\ 1,\ \text{or}\ 4 \pmod 5 \text{ and } 2(\mu+1)^2 \equiv 0,\ 2,\ \text{or}\ 3 \pmod 5$$

and hence if (8.33) is a multiple of 5, the coefficient $(2\nu+1)$ in (8.32) is also a multiple of 5, and therefore the coefficient of q^{5m+5} in

$$q\{(1-q)(1-q^2)\ldots\}^4$$

is a multiple of 5.

Next, in the binomial expansion of $\frac{1}{(1-q)^5}$, all coefficients are divisible by 5 except $1, q^5, q^{10}, \ldots$, which have residue 1 (mod 5). That is to say

(8.35) $$\frac{1}{(1-q)^5} \equiv \frac{1}{1-q^5} \pmod 5$$

or

(8.36) $$\frac{1-q^5}{(1-q)^5} \equiv 1 \pmod 5$$

where such congruences are taken to mean that all coefficients are congruent (mod 5). Hence the coefficient of q^{5m+5} in

(8.37 $$q\frac{(1-q^5)(1-q^{10})\ldots}{(1-q)(1-q^2)\ldots} = q\{(1-q)(1-q^2\ \ldots\}^4 \frac{(1-q^5)(1-q^{10})\ldots}{\{(1-q)(1-q^2)\ldots\}^5}$$

is a multiple of 5, and so therefore is that in

(8.38) $$\frac{q}{(1-q)(1-q^2)\ldots}$$

and this coefficient is $p(5m+4)$.

We can prove (8.15b) similarly, using the square of Jacobi's identity instead of the product of Euler's and Jacobi's, but there seems to be no such simple proof (8,15c).

Ramanujan went a good deal further; he proved congruences with moduli 5^2, 7^2, 11^2, and was led to a general conjecture: If $\delta = 5^a 7^b 11^c$ and

(8.39) $$24\lambda \equiv 1 \pmod{\delta}$$

then

(8.40) $$p(m\delta + \lambda) \equiv 0 \pmod{\delta}$$

for every m.

It has however been shown recently by Gupta that the conjecture is false when $\delta = 7^3$. Here $\lambda = 243$ and

(8.41) $$p(243) = 133978259344888$$

is not divisible by 7^3.

On the other hand Krezmer has proved the conjecture for $\delta = 5^3$, viz.

(8.42) $$p(125m + 99) \equiv 0 \pmod{5^3}$$

and Lehmer has found some evidence of its truth for $\delta = 11^3$. In this case $\lambda = 721$, and Lehmer, using a method which will be referred to later, finds

(8.43) $$161061755750279477635534762$$

which is divisible by 11^3, as the value of $p(721)$.

There is another proof of (8.15) which is much more difficult than the one here given, but which led Ramanujan much deeper into the theory of elliptic modular functions. In the same paper in which he proved (8.15), Ramanujan stated without proof the two remarkable identities

(8.44) $$p(4) + p(9)q + p(14)q^2 + \ldots = \frac{5\{(1-q^5)(1-q^{10})\ \ldots\}^5}{\{(1-q)(1-q^2)\ \ldots\ldots\}^6}$$

and

(8.45) $$p(5) + p(12)q + p(19)q^2 + \ldots = \frac{7\{(1-q^7)(1-q^{14})\ \ldots\}^3}{\{(1-q)(1-q^2)\ \ldots\ldots\}^4} + \frac{49q\{(1-q^7)(1-q^{14})\ \ldots\}^7}{\{(1-q)(1-q^2)\ \ldots\ldots\}^8}.$$

These make (8.15a) and (8.15b) evident, and also provide proofs of the congruences to moduli 5^2 and 7^2. Thus if we assume (8.44), we have

(8.46) $$\frac{p(4)q + p(9)q^2 + \ldots}{5\{(1-q^5)(1-q^{10})\ \ldots\}^4} = \frac{q}{(1-q)(1-q^2)\ \ldots} \frac{(1-q^5)(1-q^{10})\ldots}{\{(1-q)(1-q^2)\ldots\}^5} \equiv \frac{q}{(1-q)(1-q^2)\ldots} \pmod 5$$

Hence (after what we have proved already) the coefficients of q^{5m+5} on the left-hand side is a multiple of 5; and from this it follows that

(8.47) $$p(25m + 24) \equiv 0 \pmod{5^2}$$

Similarly (8.45) leads to

(8.48) $$p(49m + 47) \equiv 0 \pmod{7^2}$$

Ramanujan never published a complete proof of (8.44) or (8.45); but proofs have been found by Darling and Mordell. Mordell's proof is tolerably short, but demands much more knowledge of the general theory of the modular functions than Ramanujan ever possessed.

9. The Rogers-Ramanujan identities

There are two theorems, the 'Rogers-Ramanujan identities', in which Ramanujan had been anticipated by a much less famous mathematician, but which are certainly two of the most remarkable formulae which even he ever wrote down.

The Rogers-Ramanujan identities are

(9.1) $$1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \dots + \frac{q^{m^2}}{(1-q)(1-q^2)\dots(1-q^m)} + \dots = \frac{1}{(1-q)(1-q^6)\dots(1-q^4)(1-q^9)\dots}$$

and

(9.2) $$1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \dots + \frac{q^{m(m+1)}}{(1-q)(1-q^2)\dots(1-q^m)} + \dots = \frac{1}{(1-q^2)(1-q^7)\dots(1-q^3)(1-q^8)\dots}.$$

The exponents in the denominators on the right form in each case two arithmetical progressions with the difference 5. This is the surprise of the formulae; the 'basic series' on the left are of a comparatively familiar type. The formulae have a very curious history. They were found first, so long ago as 1894, by Rogers [L. J. Rogers, P.L.M.S. (1), 1894], a mathematician of great talent but comparatively little reputation, and one of whom very few people, had it not been for Ramanujan, might ever have heard. Rogers was a fine analyst, who anticipated 'Hölder's inequality' though without recognizing its importance or stating it in what is now its classical form. See Hardy, Littlewood, and Polya, Inequalities, pp. 21-26. His gifts were, on a smaller scale, not unlike Ramanujan's own; but no one paid much attention to his work, and this particular paper was quite neglected.

Ramanujan rediscovered the formulae some time before 1913, and stated them in the first of his letters to Hardy. He had then no proof (and knew that he had none); and neither Hardy nor MacMahon nor Perron could find one. They are therefore stated without proof in the second volume of MacMahon's Combinatory Analysis, published in 1916.

The mystery was solved, trebly, in 1917. In that year Ramanujan found a proof which will be given later. A little later he came accidentally across Rogers' paper and the more elaborate proof given there. Ramanujan was quite surprised by this find, and expressed the greatest admiration for Rogers' work. His rediscovery led incidentally to a belated recognition of Rogers' talent, and in particular to his election to the Royal Society. Finally I. Schur, who was then cut off from England by the war, rediscovered the identities again. Schur published two proofs [Berliner Sitzungsberichte, 1917,pp.301-321. one of which is 'combinatorial' and quite unlike any other proof known. There are now seven published proofs, three by Rogers [one in the paper already referred to, one in Proc.Camb.Phil.Soc., 19, pp.211-216, and one in P.L.M.S. (2), vol. 16 (1917), pp. 315-336, one by Ramanujan, two by Schur, and a later proof, based on quite different ideas, by Watson [J.L.M.S., vol. 4 (1929), pp. 4-9, and five at any rate of these proofs differ fundamentally. None of them is both simple and straightforward, and probably it would be unreasonable to hope for a proof which is. The simplest proofs are essentially verifications.

There are two well known identities of Euler which have a strong superficial resemblance to (9.1) and (9.2), viz.

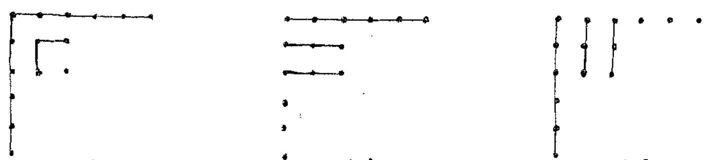
(9.3) $$1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^2)(1-q^4)} + \dots + \frac{q^{m^2}}{(1-q^2)(1-q^4)\dots(1-q^{2m})} + \dots = (1+q)(1+q^3)(1+q^5)\dots$$

and

$$(9.4)\qquad 1+\frac{q^2}{1-q^2}+\frac{q^6}{(1-q^2)(1-q^4)}+\dots+\frac{q^{m(m+1)}}{(1-q^2)(1-q^4)\dots(1-q^{2m})}=(1+q^2)(1+q^4)(1+q^6)\dots$$

(the difference being that only even powers of q occur in the denominators). These are easily proved, algebraically or arithmetically, whereas the Rogers-Ramanujan formulae lie much deeper (as is indicated, of course, by the way in which they involve the number 5). It will, however, help us to understand the formulae if we begin by considering one of the simpler formulae, say (9.3).

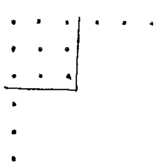
We have seen that the right hand side of (9.3) enumerates partitions into odd and unequal parts: thus $15 = 11 + 3 + 1 = 9 + 5 + 1 = 7 + 5 + 3$ has 4 such partitions. Let us take, for example, the partition $11 + 3 + 1$, and represent it graphically as in (A), the points on one broken line corresponding to a part of the partition:



(A) (B) (C)

We can also read the graph (considered as an array of points) as in (B) or (C), along a series of horizontal or vertical lines. The graphs (B) and (C) differ only in orientation, and each of them corresponds to another partition of 15, viz.: $6 + 3 + 3 + 1 + 1 + 1$. A partition like this, symmetrical about the south-easterly direction, is called by MacMahon a <u>self-conjugate</u> partition, and the graphs establish a (1-1) correspondence between self-conjugate partitions and partitions into odd and unequal parts. The right-hand side of (9.3) enumerates odd and unequal partitions, and therefore the identity will be proved if we can show that its left-hand side enumerates self-conjugate partitions.

Now our array of points may be read in a fourth way, viz. as in (D):



(D)

Here we have a square of 3^2 points, and two 'tails', each representing a partition of $\frac{1}{2}(15 - 3^2) = 3$ into 3 parts at most (and in this particular case all 1's). Generally, a self-conjugate partition of n can be read as a square of m^2 points, and two tails representing partitions of $\frac{1}{2}(n - m^2)$ into m parts at most. Given the (self-conjugate) partition, m, and the reading of the partition is fixed; conversely, given n and given any square m^2 not exceeding n, there is a group of self-conjugate partitions of n based upon a square of m^2 points.

Now the functions

$$(9.5)\qquad (a)\ \frac{1}{(1-q)(1-q^2)\dots(1-q^m)},\qquad (b)\ \frac{1}{(1-q^2)(1-q^4)\dots(1-q^{2m})},\qquad (c)\ \frac{q^{m^2}}{(1-q^2)(1-q^4)\dots(1-q^{2m})}$$

enumerate (a) partitions of n into at most m parts, (b) partitions of n into at most m even parts (or of $\frac{1}{2}n$ into at most m parts of any kind), and (c) partitions of $\frac{1}{2}(n-m^2)$ into at most m parts; and each of the last partitions corresponds, as we have seen, to a self-conjugate partition of n. Hence the left-hand side of (9.3) which is obtained by summing (c) with respect to m enumerates the self-conjugate partitions of n, and thus proves (9.3).

Franklin's proof of Euler's identity (8.25), referred to above, is another good example of this kind of reasoning.

Suppose that

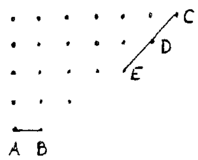
$$(9.6)\qquad (1-q)(1-q^2)(1-q^3)\dots = \sum c_n q^n$$

A partition of n into μ unequal parts contributes $(-1)^{\mu}$ to the coefficient c_n, so that

$$(9.7)\qquad c_n = p_2(n) - p_1(n)$$

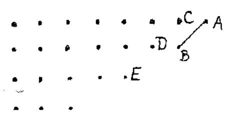
where $p_2(n)$ and $p_1(n)$ are the numbers of partitions into an even and an odd number of unequal parts. We try to establish a (1, 1) correspondence between partitions of the two types. The correspondence cannot be exact, since c_n is not always 0.

We take a graph E representing a partition of n into any number of unequal parts, in descending order. We call the lowest line AB the 'base' β of the graph. From C, the extreme north-east node, we draw the longest south-westerly line possible in the graph; it might of course contain one node only. This line CDE we call the 'slope' σ of the graph. We write $\beta < \sigma$ when (as in graph E) there are more nodes in σ than in β and use a similar notation in other cases. Then there are three possibilities:



(E)

(a) $\beta < \sigma$ We move β into a position parallel to and outside σ as shown in graph (F). This gives a new partition into decreasing unequal parts, and into a number of such parts whose parity is opposite to that of the number in (E). We call this operation O, and the converse operation (removing σ and placing it below β) Ω. It is plain that Ω is not possible when $\beta < \sigma$ without violating the conditions of the graph.



(F)

(b) $\beta = \sigma$ In this case O is possible (as in graph G) unless β meets σ (as in graph H), when it is impossible. Ω is not possible in either case.

(c) $\beta > \sigma$ In this case O is always impossible. Ω is possible in graph (K) unless β meets σ and $\beta = \sigma + 1$ (as in graph L).

(G) (H) (K) (L)

Ω is impossible in the last case because it would lead to a partition with two equal parts.

To sum up, there is a (1, 1) correspondence between the two types of partitions except in the cases exemplified by (H) and (L). In the first of these exceptional cases n is of the form

$$k + (k+1) + (k+2) + \dots + (2k-1) = \tfrac{1}{2}(3k^2-k)$$

and in this case there is an excess of one even or one odd partition according as k is even or odd. In the second case n is of the form

$$(k+1) + (k+2) + \dots + 2k = \tfrac{1}{2}(3k^2+k)$$

and the excess is the same. Hence c_n is 0 unless $n = \tfrac{1}{2}(3k^2 \pm k)$, when $c_n = (-1)^k$. This is Euler's theorem.

The algebraic proof of (9.3) is less prolix than the combinatorial proof, but also less illuminating. It depends upon Euler's device of the introduction of a second parameter. We write

(9.8) $$f(a) = (1+aq)(1+aq^3)(1+aq^5)\dots = 1 + c_1a + c_2a^2 + c_3a^3 + \dots$$

where the coefficients are functions of q. Then

(9.9) $$f(a) = (1+aq)f(aq^2)$$

and so, by equating coefficients

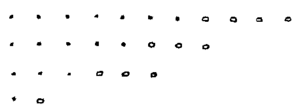
(9.10) $$c_1 = q + c_1q^2, \qquad c_2 = c_1q^3 + c_2q^4, \dots, \qquad c_m = c_{m-1}q^{2m-1} + c_mq^{2m}, \dots$$

(9.11) $$c_m = \frac{q^{1+3+5+\dots+(2m-1)}}{(1-q^2)(1-q^4)\dots(1-q^{2m})} = \frac{q^{m^2}}{(1-q^2)(1-q^4)\dots(1-q^{2m})}$$

$$(9.12) \qquad (1+aq)(1+aq^3)(1+aq^5)\ldots = 1 + \frac{aq}{1-q^2} + \frac{a^2q^4}{(1-q^2)(1-q^4)} \ldots$$

and (9.3) and (9.4) are the special cases $a = 1$ and $a = q$. Any number of examples of similar reasoning may be found in MacMahon's book or in Bailey's tract.

We return now to the first of the Rogers-Ramanujan formulae. We can exhibit a square m^2 as $1 + 3 + 5 + \ldots + (2m-1)$ or in the manner shown by the black dots of (M). If we now take any partition of $n-m^2$ into m parts at most, with the parts in descending order, and add it to the graph as shown by the circles of (M) where $m = 4$ and $n = 4^2 + 11 = 27$ we obtain a partition of n (here $27 = 11 + 8 + 6 + 2$) into parts without repetitions or sequences or <u>parts whose minimal difference is 2</u>. The left-hand side of (9.1) enumerates this type of partition of n.



(M)

On the other hand the right-hand side enumerates partitions into numbers $5m+1$ and $5m+4$. Hence (9.1) may be restated as a 'combinatorial' theorem: <u>the number of partitions of n with minimal difference 2 is equal to the number of partitions into parts $5m+1$ and $5m+4$</u>. Thus, when $n = 9$ there are 5 partitions of each type 9, 8+1, 7+2, 6+3, 5+3+1 of the first kind, and 9, 6+1+1+1, 4+4+1, 4+1+1+1+1+1, 1+1+1+1+1+1+1+1+1 of the second. There is a similar combinatorial interpretation of (9.2).

These forms of the theorems are MacMahon's (or Schur's); neither Rogers nor Ramanujan ever considered their combinatorial aspect. It is natural to ask for a proof in which, as in our first proof of (9.3), we set up a direct (1, 1) correspondence between the two sets of partitions, but no such proof is known. Schur's 'combinatorial' proof is based not on (9.1) itself, but on a transformation of the formula, with each side multiplied by $(1-q)(1-q^2)(1-q^3) \ldots$ and is more like Franklin's proof of (9.3).

Ramanujan's proof is as follows. We can write the right-hand side of (9.1) as

$$(9.13) \qquad \frac{1}{\prod\{(1-q^{5m+1})(1-q^{5m+4})\}} = \frac{\prod\{(1-q^{5m})(1-q^{5m+2})(1-q^{5m+3})\}}{(1-q^m)}$$

and the numerator on the right can be transformed, by Jacobi's formula (8.16) replacing q, z by $q^{\frac{5}{2}}$, $q^{\frac{1}{4}}$, into

$$(9.14) \qquad 1 - q^2 - q^3 + q^9 + q^{11} \ldots$$

where the indices are the numbers $\frac{1}{2}(5n^2 \pm n)$. We have therefore to prove that

$$(9.15) \qquad 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \ldots = \frac{1-q^2-q^3+q^9+q^{11}\ldots}{(1-q)(1-q^2)(1-q^3)\ldots}$$

We now introduce another variable, as in our second proof of (9.3), but in a much more sophisticated way. We write

$$(9.16) \qquad F(a) = 1 + \frac{aq}{1-q} + \frac{a^2q^4}{(1-q)(1-q^2)} + \frac{a^3q^9}{(1-q)(1-q^2)(1-q^3)} + \ldots$$

and

$$(9.17) \qquad G(a) = 1 - a^2q^2(1-aq^2)\frac{1}{1-q} + \frac{a^4q^9(1-aq^4)(1-aq)}{(1-q)(1-q^2)} + \ldots$$
$$+ (-1)^n a^{2n} q^{\frac{1}{2}(5n^2-n)}(1-aq^{2n})\frac{(1-aq)(1-aq^2)\ldots)1-aq^{n-1})}{(1-q)(1-q^2)\ldots(1-q^n)} + \ldots$$

The choice of $F(a)$ is natural enough, but that of $G(a)$ is, of course, dictated by our knowledge of what we want to prove.

It may now be verified that $G(a)$ satisfies

$$(9.18) \qquad G(a) = (1-aq)G(aq) + aq(1-aq)(1-aq^2)G(aq^2)$$

so that

$$H(a) = \frac{G(a)}{(1-aq)(1-aq^2)\ldots} \tag{9.19}$$

satisfies

$$H(a) = H(aq) + aqH(aq^2). \tag{9.20}$$

Also

$$F(a) - F(aq) = \frac{aq^2(1-q)}{(1-q)} + \frac{a^2q^6(1-q^2)}{(1-q)(1-q^2)} + \ldots = aqF(aq^2) \tag{9.21}$$

$$F(a) = F(aq) + aqH(aq^2).$$

Since $F(a)$ and $H(a)$ are power series, and their first few terms are the same, it follows that

$$F(a) = H(a) = \frac{G(a)}{(1-aq)(1-aq^2)\ldots} \tag{9.22}$$

and this reduces to (9.1) when $a = 1$. Similarly (9.2) follows from (9.22) when $a = q$.

This proof may be called a 'simple' proof: it is short, and can be followed by any mathematical student with fair technical skill; but it is very unilluminating.

The formulae which have been written down have further remarkable consequences. From

$$F(a) = F(aq) + aqF(aq^2) \tag{9.23}$$

we deduce

$$\frac{F(a)}{F(aq)} = 1 + \frac{aqF(aq^2)}{F(aq)} = 1 + \frac{aq}{1+\frac{F(aq)}{F(aq^2)}} = 1 + \frac{aq}{1+}\,\frac{aq^2}{1+}\,\frac{aq^3}{1+}\ldots . \tag{9.24}$$

In particular

$$1 + \frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\ldots = \frac{F(1)}{F(q)} = \frac{(1-q)(1-q^6)\ldots(1-q^4)(1-q^9)\ldots}{(1-q^2)(1-q^7)\ldots(1-q^3)(1-q^8)\ldots} = \frac{1-q-q^4+q^7+q^{13}\ldots}{1-q^2-q^3+q^9+q^{11}\ldots} \tag{9.25}$$

is a quotient of elliptic theta functions. This formula is the key to Ramanujan's evaluation of the continued fraction for special values of q and in particular to the extraordinary formulae

$$\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\ldots = \left\{\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)} - \frac{\sqrt{5}+1}{2}\right\} e^{\frac{2\pi}{5}} \tag{9.26}$$

and

$$\frac{1}{1+}\,\frac{e^{-2\pi\sqrt{5}}}{1+}\,\frac{e^{-4\pi\sqrt{5}}}{1+}\ldots = \left[\frac{\sqrt{5}}{1+\sqrt[5]{5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}} - 1}} - \frac{\sqrt{5}+1}{2}\right] e^{\frac{2\pi}{\sqrt{5}}} \tag{9.27}$$

which he sent to Professor Hardy in 1913.

10. Asymptotic Properties of $p(n)$.

In §§ 8 and 9 we have considered arithmetic and combinatorial properties of partitions. We now turn our attention to a question of a very different type from those previously discussed. We ask '*How big is $p(n)$ for large n?*' Hardy and Ramanujan were the first to study this question. It was of particular interest, first because it was a really new type of problem, and second because their solution was so surprisingly successful.

There is one obvious method of attack on any such problem (if no strictly elementary method suggests itself). If a_n is positive, and the power series

$$F(x) = \sum a_n x^n \tag{10.1}$$

has a radius of convergence 1, then there is a general correspondence between the order of magnitude of of a_n, for large n, and that of $F(x)$, for x near 1. The first step, therefore, is to determine the order of magnitude of Euler's function

$$F(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots} \tag{10.2}$$

when $x \to 1$. (We use the letter x now instead of q, q being wanted for other purposes.) This is quite simple, if we are content with a rough approximation, since

$$(10.3) \qquad \log F(x) = \sum_n \log \frac{1}{1-x^n} = \sum_{m,n} \frac{x^{mn}}{m} = \sum_m \frac{x^m}{m(1-x^m)}$$

and

$$(10.4) \qquad \sum_m \frac{x^m}{m(1-x^m)} \sim \frac{1}{1-x} \sum_m \frac{1}{m^2} = \frac{\pi^2}{6(1-x)}$$

It follows that the order of magnitude of $F(x)$ is, to a first approximation, that of

$$(10.5) \qquad \exp \frac{\pi^2}{6(1-x)}$$

We want to know the order of a_n corresponding to this order for $F(x) = \sum a_n x^n$

If $a_n = n^\alpha$, where $\alpha > -1$, and $x = e^{-y}$ so that $y \to 0$ as $x \to 1$, then

$$(10.6) \qquad F(x) = \sum n^\alpha x^n = \sum n^\alpha e^{-ny} \sim \int_0^\infty t^\alpha e^{-ty}\, dt$$

$$= \frac{\Gamma(\alpha+1)}{y^{\alpha+1}} \sim \frac{\Gamma(\alpha+1)}{(1-x)^{\alpha+1}}$$

On the other hand, if a_n were as large as $e^{\delta n}$ for some positive δ, then the series would diverge before x reaches 1. It is plain that a_n must be smaller than this, but larger than any power of n. It is natural to conjecture that the right order is about

$$e^{Bn^b}$$

for some b between 0 and 1 and some B.

The order of

$$(10.7) \qquad G(x) = \sum e^{Bn^b} x^n = \sum e^{Bn^b - ny}$$

may be calculated roughly from that of its maximum term. This occurs when $B\,b\,n^{b-1} = y$, approximately; and the maximum term is then about

$$(10.8) \qquad \exp\left\{C(1-x)^{\frac{-b}{1-b}}\right\}$$

where

$$(10.9) \qquad C = B^{\frac{1}{1-b}}\, b^{\frac{b}{1-b}}\, (1-b)$$

This agrees with (10.5) if $b = \frac{1}{2}$ and $B^2 = \frac{2\pi^2}{3}$ and we conclude that the order of $p(n)$ should be about

$$(10.10) \qquad e^{A\sqrt{n}}$$

where $A = \pi\sqrt{\frac{2}{3}}$

We can put these inferences in order if we do not ask for too precise a result. We can prove, by arguments of the so-called 'Tauberian' kind, that

$$(10.11) \qquad p(n) = \exp\left[\{A + o(1)\}\sqrt{n}\right]$$

or that

$$(10.12) \qquad \log p(n) \sim A\sqrt{n}$$

an asymptotic formula, in the ordinary sense, for $\log p(n)$. This however is a very crude answer to our question with which we cannot possibly be content.

We want, at the least, an asymptotic formula for $p(n)$ itself. Actually

$$(10.13) \qquad p(n) \sim \frac{1}{4n\sqrt{3}}\, e^{A\sqrt{n}}$$

but we cannot hope to get so far as this by arguments of so elementary and general a kind. They are naturally effective, so far as they go, over a wide range of problems. Thus we can prove (assuming the prime number theorem) that the number of partitions of n into primes is

$$\exp\left\{\frac{2\pi}{\sqrt{3}}\sqrt{\frac{n}{\log n}}\right\} \tag{10.14}$$

with the same degree of accuracy as (10.11).

Our natural resource is Cauchy's theorem. This tells us that

$$p(n) = \frac{1}{2\pi i}\int_C \frac{F(x)}{x^{n+1}}\,dx \tag{10.15}$$

where C is a contour around the origin. We must move C into the most advantageous position and study the integral directly. There is of course nothing in the least novel in this idea, which is that which dominates the whole analytic theory of numbers, and in particular the theory of primes; but the setting is different and it is instructive to compare the two problems.

In the theory of primes our generating functions were Dirichlet's series $\sum a_n n^{-s}$, and the proof of the Prime Number Theorem depended upon the integral

$$\psi^*(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta'(s)}{\zeta(s)}\,\frac{x^s}{s}\,ds \tag{10.16}$$

We moved the contour of integration to the left, across the pole at $s = 1$, and inferred that $\psi^*(x)$ and $\psi(x)$ differed from the residue x at the pole by an error of smaller order than x.

The conclusion was correct, but the argument was difficult to justify because $\zeta(s)$ behaves in a very complicated way at infinity. In particular the location of its zeros is still highly mysterious. On the other hand, there is no difficulty at all about the critical singularity which yields the dominant term, a pole at $s = 1$ of the simplest possible character.

The singularities of the $F(x)$ of our present problem are very much more complicated. They cover the unit circle $|x| = 1$. The circle is a 'barrier' for the function, which does not exist outside it, and there can be no question of 'moving C across the singularities." All that we can hope to do is to move C <u>close to</u> the singularities and study each part of it in detail.

For all this, however, there are strong consolations. The function $F(x)$ is one of a well-known class, the elliptic modular functions, whose properties have been studied intensively and are very exactly known. These functions all have the same peculiarities as $F(x)$, and exist only inside the circle; but they satisfy remarkable functional equations which enable us to determine their behaviour near any point of the circle, very precisely. In particular, $F(x)$ satisfies the equation

$$F(x) = \frac{x^{1/24}}{\sqrt{2n}}\sqrt{\log 1/n}\;.\;\exp\left\{\frac{\pi^2}{6\log(1/x)}\right\}F(x') \tag{10.17}$$

where

$$\log 1/x \;\; \log 1/x' = 4\pi^2, \quad x' = \exp\left\{\frac{-4\pi^2}{\log(1/x)}\right\} \tag{10.18}$$

If, for example, x is positive and near to 1, the x' is extravagantly small and $F(x')$ is practically 1; so that (10.17) expresses $F(x)$ effectively, in terms of elementary functions. There are similar formulae associated with other points of the circle, such as

$$-1,\; e^{\frac{2\pi i}{3}},\; e^{-2\pi i/3},\; i,\; -i,\; e^{2\pi i/5},\; \ldots$$

(generally, with all roots of unity); but (10.17) alone is enough to enable us to make great progress.

In particular, if we take C in (10.15) to be a circle with just the right radius, which turns out to be $1 - 1/n$, we can substitute from (10.17) into (10.15) and neglect $F(x')$, with an error which turns out to be of order

$$e^{H\sqrt{n}}$$

where

$$H < A = \pi\sqrt{2/3} \quad .$$

There are then only elementary functions in the integral, and we can calculate it very precisely.

The result is the formula

$$p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn}\left(\frac{e^{A\lambda_n}}{\lambda_n}\right) + O(e^{H\sqrt{n}}) \tag{10.19}$$

where

$$\lambda_n = \sqrt{n - 1/24} \quad , \qquad H < A. \tag{10.20}$$

This includes (10.13) and is very much more precise. The form of the dominant term is at first sight rather mysterious but it arises naturally from the analysis. In particular the '$-1/24$' in λ_n arises naturally from the index $1/24$ in (10.17).

The formula (10.19) was found independently by Uspensky, Bulletin de l'Academie des Sciences de l'U. S. S. R., series 6, vol. 14 (1920), pp. 199 - 218. Uspensky's paper was published a little after the Hardy-Ramanujan paper, which developed the formula further, so that his proof, in some ways rather simpler than that Hardy and Ramanujan, has been less noticed than it deserves.

The formula (10.19) is by no means the end of the matter. The formulae (10.17) and (10.18) are those appropriate for the study of $F(x)$ near $x = 1$. There are, as has been remarked above, similar formulae associated with other 'rational points'

$$x_{p,q} = e^{2p\pi i/q} \tag{10.21}$$

on the unit circle. One may say (naturally very roughly) that these 'rational singularities' are the heaviest singularities of $F(x)$, that $F(x)$ is bigger near them than near other points of the circle, and that their contributions to the integral (10.15) may be expected to outweigh those of other points.

Further, these rational singularities diminish in weight as q increases. When $x \to 1$ along a radius, $F(x)$ behaves roughly like

$$\exp \frac{\pi^2}{6(1-x)}$$

while when $x \to x_{p,q}$, it behaves more like

$$\exp \frac{\pi^2}{6\, q^2\, (1-|x|)}$$

It is reasonable to expect that

$$p(n) = P_1(n) + P_2(n) + \dots + P_Q(n) + R(n) \tag{10.22}$$

where $P_1(n)$ is the dominant term in (10.19), $P_2(n)$, $P_3(n)$, ... , $P_Q(n)$ are similar in form, but with smaller numbers A_2, A_3, ... , A_Q in the place of A, and $R(n)$ is an error of lower order than $e^{A_Q\sqrt{n}}$.

The proof of all this can be put through without much additional difficulty. The form of $P_q(n)$ is

$$P_q(n) = L_q(n)\ \phi_q(n) \tag{10.23}$$

where

$$\phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}} \cdot \frac{d}{dn}\left(\frac{e^{A\lambda_n/q}}{\lambda_n}\right) \tag{10.24}$$

$$L_q(n) = \sum_p \omega_{p,q}\, e^{-\frac{2n\pi i p}{q}}$$

where $0 < p \leq q$ $\quad (p,q) = 1$ and $\omega_{p,q}$ is a certain 24q-th root of unity. Alternative expressions for $\omega_{p,q}$ are by Hardy and Ramanujan, Proceedings of the London Mathematical Society, series 2, vol. 17, (1918), pp. 75 - 115, and Rademacher (to be published in the near future). Thus

(10.25) $$L_1(n) = 1 \qquad \phi_1(n) = P_1(n) , \quad A_Q = A/Q$$

and

(10.26) $$R(n) = O(e^{H_Q\sqrt{n}})$$

where

(10.27) $$H_Q < A_Q = A/Q$$

(so that $H_Q \to 0$ when $Q \to \infty$). We can thus find $p(n)$ with error $O(e^{\delta\sqrt{n}})$ and an arbitrarily small positive δ.

At this point Hardy and Ramanujan might have stopped had it not been for Major Macmahon's love of calculation. Macmahon was a practised and enthusiastic computer, and made a table of $p(n)$ up to $n = 200$. In particular he found that

(10.28) $$p(200) = 3972999029388$$

and Hardy and Ramanujan naturally took this value as a test for their asymptotic formula. They expected a good result, with an error of perhaps one or two figures, but had never dared to hope for such a result as they found. Actually 8 terms of their formulae gave $p(200)$ with an error of .004. They were inevitably led to ask whether the formula could not be used to calculate $p(n)$ exactly for any large n.

It is plain that, if this is possible, it will be necessary to use a 'large' number of terms of the series, that is to say to make Q a function of n. The final result is as follows. There are constants α, K such that

(10.29) $$p(n) = \sum_{q < \alpha\sqrt{n}} P_q(n) \quad + R(n) ,$$

where

(10.30) $$|R(n)| < K n^{-1/4}$$

and, since $p(n)$ is an integer, (10.29) will give its value exactly for sufficiently large n. This formula seems to be one of the rare formulas which are both asymptotic and exact; it tells us all we want to know about the order and approximate form of $p(n)$, and appears also to be adapted for exact calculation. It was, in fact, from this formula that Lehmer calculated the value of $p(721)$.

But at this point it was necessary, until very recently, to make a curious reservation. The values of $p(200)$ and $p(243)$ were known, because they had been calculated directly by MacMahon and Gupta, but calculations based upon (10.29) were not decisive. We cannot use the formula to prove that $p(721)$ has a particular value until we have found numerical values for α and K; Hardy and Ramanujan had merely proved their existence. It was necessary to go over all their analysis, give numerical values to all their 'constants', and replace all their 'O' terms by terms with numerical bounds. In the meantime Lehmer's calculations, which used 21 terms of the series, led to the value

(10.31) $$161061755750279477635534762.0041$$

gave a very strong presumption about the value of $p(721)$ but were not conclusive.

The gap has now been filled by Rademacher, who, in an attempt to simplify the Hardy-Ramanujan analysis, was led to make a very fortunate formal change. Hardy and Ramanujan worked, not exactly with the function

(10.32) $$\phi_1(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} \left(\frac{e^{A\lambda_n}}{\lambda_n} \right)$$

but with the 'nearly equivalent' function

$$(10.33) \qquad \frac{1}{\pi\sqrt{2}}\left(\frac{\cosh A\lambda_n - 1}{\lambda_n}\right)$$

(afterwards discarding the less important parts of the function). Rademacher worked with

$$(10.34) \qquad \psi_1(n) = \frac{1}{\pi\sqrt{2}}\,\frac{d}{dn}\left(\frac{\sinh A\lambda_n}{\lambda_n}\right)$$

which is also 'nearly equivalent'; and this apparently slight change has a very important effect, since it leads to an identity for p(n).

The function

$$(10.35) \qquad \psi_q(n) = \frac{\sqrt{q}}{\pi\sqrt{2}}\,\frac{d}{dn}\left(\frac{\sinh\,(A\,\lambda_n/q)}{\lambda_n}\right)$$

behaves for fixed n and large q, like a multiple of $q^{1/2}.\ q^{-3} = q^{-5/2}$, and $|L_q(n)| \leqq q$, so that

$$(10.36) \qquad \sum L_q(n)\ \psi_q(n)$$

is convergent. The convergence of $\sum L_q(n)\ \phi_q(n)$ still remains in doubt.* Rademacher proves that

$$(10.37) \qquad p(n) = \sum_{q=1}^{\infty} L_q(n)\ \psi_q(n)$$

and that the remainder after Q terms is less than

$$(10.38) \qquad \frac{44\,\pi^2}{225\,\sqrt{3}}\,Q^{-1/2} + \frac{\pi\sqrt{2}}{75}\left(\frac{Q}{n-1}\right)^{\frac{1}{2}}\sinh\ \pi\sqrt{2/3}\ \frac{\sqrt{n}}{Q}$$

This is of order $n^{-1/4}$ when Q is of order n, as in the older work.

It is now possible to justify the results of Lehmer's calculations:

(10.39)

q	$P_q(721)$
1	161061755750279601828302117.84821
2	- 124192062781.96844
3	- 706763.61926
4	2169.16829
5	0.00000
6	14.20704
7	6.07827
8	0.18926
9	0.04914
10	0.00000
11	0.08814
12	- 0.03525
13	0.03247
14	- 0.00687
15	0.00000
16	- 0.01133
17	0.00000
18	- 0.00553
19	0.00859
20	0.00000
21	- 0.00524

p(721) = 161061755750279477635534762.0041

* D.H.Lehmer has recently shown that this series does not converge. His result is to be published in the Proc.L.M.S.

In this case n = 721, Q = 21. There is a small addition to the possible error (10.38) because Lehmer works with $\phi_q(n)$ instead of $\psi_q(n)$, but it is easy to obtain the upper bound 0.37819. Any bound less than $121 = 11^2$ would suffice since

(10.40) $$p(721) \equiv 0 \pmod{11^2}$$

and the error, if any, is a multiple of 121.

11. The function $c_q(n)$.

Ramanujan was greatly interested in two functions, $c_q(n)$ and $\tau(n)$, which arose from his studies on the arithmetical theory of elliptic functions, especially in their application to the problem of representing numbers as sums of squares.

Ramanujan defined

(11.1) $$c_q(n) = \sum_{\substack{0<p\le q\\(p,q)=1}} e^{-2np\pi i/q}$$

If in this sum we combine the terms for p and q-p we obtain

(11.2) $$c_q(n) = \sum \cos 2np\pi/q$$

whence it is evident that $c_q(n)$ is real. Moreover

(11.3) $$c_q(n) = \sum_{\rho} \rho^n$$

where ρ ranges over the primitive q^{th} roots of unity, Laundau (Primzahlen II p. 572) and Jensen (Proc. 3 Skand. Kongress pp. 145 - 147) had considered the function $c_q(n)$ for n = 1, but Ramanujan was the first to introduce the parameter n.

Aside from the three essentially equivalent forms of $c_q(n)$ in (11.1, 2, 3) we shall prove that

(11.4) $$c_q(n) = \sum_{\delta|(q,n)} \delta\,\mu\left(\frac{q}{\delta}\right)$$

and in particular

$$c_q(1) = \mu(q),$$

where the μ is the familiar Möbius function. From (11.4) we see at once that $c_q(n)$ is bounded if either of q or n is bounded.

Ramanujan's proof of (11.4) is very simple. Let

(11.5) $$\eta_q(n) = \sum_{s=0}^{q-1} e^{-2ns\pi i/q} = \begin{cases} q & \text{if } q \mid n \\ 0 & \text{if } q \nmid n \end{cases}$$

Then

(11.6) $$\eta_q(n) = \sum_{\delta|q} c_\delta(n)$$

since every q^{th} root of unity is a primitive δth root of unity where δ is an appropriate divisor of q, and conversely. He now appeals to the Möbius inversion formula

(11.7) $$g(q) = \sum_{d|q} f(d) \rightleftarrows f(q) = \sum_{d|q} g(d)\,\mu\left(\frac{q}{d}\right)$$

If we take c_q as f, then η_q is g and

(11.8) $$c_q(n) = \sum_{\delta|q} \mu\left(\frac{n}{\delta}\right)\eta_\delta(n)$$

But from (11.5) $\eta_\delta(n)$ is zero unless $\delta \mid n$ when $\eta_\delta(n)$ is δ. Hence in (11.8) we may replace $\eta_\delta(n)$ by δ if we sum δ only over divisors of n. Hence

(11.9) $$c_q(n) = \sum_{\delta|q,\,\delta|n} \mu\left(\frac{n}{\delta}\right)\delta$$

which proves (11.4). We note that if $\sigma(n)$ is the sum of the divisors of n, we have

(11.10) $$\sigma(n) = \sum_{\delta | n} \delta$$

whence, comparing with (11.9), we have

(11.11) $$|c_q(n)| \leq \sigma(n)$$

and it is again evident that

(11.12) $$c_q(n) = O(1) \quad \text{when } q \longrightarrow \infty$$

Before passing on to the applications of $c_q(n)$, it is well to examine the Möbius inversion formula (11.7) more closely. It is an arithmetical paraphrase of an analytic identity. Let

(11.13) $$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \qquad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

Then

(11.14) $$F(s)\,\zeta(s) = \sum \frac{f(m)}{m^s} \sum \frac{1}{m'^s}$$

whence from $g(n) = \sum_{d|n} f(d)$ we have

(11.15) $$F(s)\,\zeta(s) = G(s)$$

and so

(11.16) $$F(s) = \frac{G(s)}{\zeta(s)} = \sum \frac{g(m)}{m^s} \sum \frac{\mu(m')}{m'^s}$$

which is paraphrased as

(11.17) $$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d)$$

But this analytic proof is unnecessarily advanced, and moreover, the validity of (11.7) has nothing to do with the convergence of F(s) or G(s). We may prove (11.7) directly. If

(11.18) $$n = p_1^{a_1} \; p_2^{a_2} \; \cdots \; p_k^{a_k}$$

then as $\mu(m) = 0$ if m has any squared factor

(11.19) $$\begin{aligned} \sum_{d|n} \mu(d) &= \mu(1) + \sum_i \mu(p_i) + \sum_{i \neq j} \mu(p_i p_j) \cdots \\ &= 1 - k + \binom{k}{2} - \binom{k}{3} \cdots \\ &= 1 \quad \text{if } n = 1 \\ &= (1-1)^k = 0 \text{ if } n > 1 \end{aligned}$$

or

(11.20) $$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

If we now assume

(11.21) $$g(n) = \sum_{d|n} f(d)$$

then

(11.22) $$\sum_{d|n} \mu(d)\, g(n/d) = \sum_{d|n} \mu(d) \sum_{c|\frac{n}{d}} f(c)$$

after substitutions. We now rewrite

(11.23) $$\begin{aligned} \sum_{d|n} \mu(d) \sum_{c|\frac{n}{d}} f(c) &= \sum_{cd|n} \mu(d)\, f(c) \\ &= \sum_{c|n} f(c) \sum_{d|\frac{n}{c}} \mu(d) \end{aligned}$$

But the second sum is zero unless n/c = 1, whence the entire sum reduces to $f(n)\,\mu(1) = f(n)$.

Hence

(11.24) $$\sum_{d \mid n} \mu(d) \; g(n/d) = f(n)$$

proving half of (11.7). As every step of this argument is reversible (11.21) must be deducible from (11.24). This completes a direct proof of (11.7).

There is another interesting proof of (11.8), in which we prove first that $c_q(n)$ is a multiplicative function of q, that is

(11.25) $$c_{qq'}(n) = c_q(n) \; c_{q'}(n) \qquad \text{if} \quad (q,q') = 1$$

and we need calculate $c_q(n)$ only when q is a power of a prime. To prove (11.25) we write

(11.26) $$\begin{aligned} c_q(n) \; c_{q'}(n) &= \sum e^{2np\pi i/q} \sum e^{2np'\pi i/q'} \\ &= \sum e^{2nP\pi i/qq'} \end{aligned}$$

where $P = pq' + p'q$. Now as p runs over a complete set of residues prime to q and p' over a complete set of residues prime to q', it is easy to show that P runs over a complete set of residues mod qq'. For P takes on $\varphi(q)\,\varphi(q') = \varphi(qq')$ values and they are all distinct and relatively prime to qq'. Hence

(11.27) $$c_{qq'}(n) = \sum e^{2nP\pi i/qq'} = c_q(n) \; c_{q'}(n).$$

Similarly if $C_q(n) = \sum_{\delta \mid n,\, \delta \mid q} \delta\, \mu\!\left(\frac{q}{\delta}\right)$ then $C_q(n)$ is a multiplicative function of q. For

(11.28) $$C_q(n) \; C_{q'}(n) = \sum_{\delta \mid n,\, \delta \mid q} \delta\, \mu\!\left(\frac{q}{\delta}\right) \sum_{\delta' \mid n,\, \delta' \mid q'} \delta'\, \mu\!\left(\frac{q'}{\delta'}\right)$$

Now as $(q, q') = 1$, à fortiori $\left(\frac{q}{\delta}, \frac{q'}{\delta'}\right) = 1$ and $\mu\!\left(\frac{q}{\delta}\right) \mu\!\left(\frac{q'}{\delta'}\right) = \mu\!\left(\frac{q}{\delta}\frac{q'}{\delta'}\right)$. Thus

(11.29) $$\begin{aligned} C_q(n) \; C_{q'}(n) &= \sum_{\delta \mid n,\, \delta \mid q,\, \delta' \mid n,\, \delta' \mid q'} \delta\delta'\, \mu\!\left(\frac{qq'}{\delta\delta'}\right) \\ &= \sum_{D \mid n,\, D \mid qq'} D\, \mu\!\left(\frac{qq'}{D}\right) \\ &= C_{qq'}(n) \end{aligned}$$

Hence to prove (11.8) we need now show only that $c_q(n) = C_q(n)$ when q is a power of a prime, say $q = \varpi^k$. Now

(11.30) $$c_{\varpi^k}(n) = \sum_p e^{2np\pi i/\varpi^k}$$

where $p = \varpi^{k-1} z + p' \qquad z = 0, 1, \dots \varpi - 1 \qquad p' \le \varpi^{k-1} \quad (p', \varpi^{k-1}) = 1$

Hence $c_{\varpi^k}(n) = \sum_{z, p'} e^{2zn\pi i/\varpi + 2np'\pi i/\varpi^k}$

where the sum over z is zero except when $\varpi \mid n$ when the sum over z is ϖ. Hence

$$c_{\varpi^k}(n) = \begin{cases} 0 & \text{if } \varpi \nmid n \\ \varpi\, c_{\varpi^{k-1}}\!\left(\frac{n}{\varpi}\right) & \text{if } \varpi \mid n \end{cases}$$

Similarly $$\begin{aligned} c_\varpi(n) &= -1 \qquad (\varpi \nmid n) \\ &= \varpi - 1 \qquad (\varpi \mid n) \end{aligned}$$

Hence generally

(11.31) $$\begin{aligned} c_{\varpi^k}(n) &= 0 \quad \text{if } \varpi^{k-1} \nmid n \\ &= -\varpi^{k-1} \quad \text{if } \varpi^{k-1} \mid n,\ \varpi^k \nmid n \\ &= \varpi^{k-1}(\varpi - 1) \quad \text{if } \varpi^k \mid n \end{aligned}$$

Moreover

(11.32) $$C_{\varpi^k}(n) = \sum_{\delta \mid (n, \varpi^k)} \delta\, \mu\!\left(\frac{\varpi^k}{\delta}\right)$$

Let ϖ^a be the highest power of ϖ dividing n, for which we shall use the notation

(11.33) $\varpi^a \| n$

Then in (11.32) $\delta = 1, \varpi, \cdots \varpi^a$ if $a \leqq k-1$, $\delta = 1, \varpi, \ldots, \varpi^k$ if $a \geqq k$.

Hence if $a \leqq k - 2$

(11.34) $C_{\varpi^k}(n) = 0$ since every μ is 0

If $a = k-1$ all μ's vanish except for $\delta = \varpi^{k-1}$ where $\mu(\varpi) = -1$ and

(11.35) $C_{\varpi^k}(n) = -\varpi^{k-1}$

If $a \geqq k$ we have $\mu(\varpi) = -1$, $\mu(1) = 1$ whence

(11.36) $C_{\varpi^k}(n) = -\varpi^{k-1} + \varpi^k = \varpi^{k-1}(\varpi - 1)$

and we see that for $q = \varpi^k$, we have

(11.37) $c_q(n) = C_q(n)$

and hence we have proved (11.8).

12. <u>The singular series and representation by squares</u>. If we form the Dirichlet series

(12.1) $$\sum_{q=1}^{\infty} \frac{c_q(n)}{q^s}$$

we see that it is absolutely convergent for $\sigma > 1$, since $c_q(n)$ is real and is bounded as $q \to \infty$. We make use of the multiplicative property of $c_q(n)$ in order to express the series as a product

(12.2) $$\sum_{q=1}^{\infty} \frac{c_q(n)}{q^s} = \prod_{\varpi} \chi_{\varpi}$$

where

(12.3) $$\chi_{\varpi} = 1 + \frac{c_{\varpi}(n)}{\varpi^s} + \frac{c_{\varpi^2}(n)}{\varpi^{2s}} + \cdots$$

Now suppose that

(12.4) $\varpi^a \| n$

According to the value of a we have

(12.5) $$a = 0 \qquad \chi_{\varpi} = 1 - \varpi^{-s}$$

$$a \geqq 1 \qquad \chi_{\varpi} = 1 + \frac{\varpi - 1}{\varpi^s} + \frac{\varpi(\varpi-1)}{\varpi^{2s}} + \cdots \frac{\varpi^{a-1}(\varpi-1)}{\varpi^{as}} - \frac{\varpi^a}{\varpi^{(a+1)s}}$$

$$= (1 - \varpi^{-s})\left(\frac{1 - \varpi^{(a+1)(1-s)}}{1 - \varpi^{1-s}}\right)$$

Hence, substituting in (12.2)

(12.6) $$\sum \frac{c_q(n)}{q^s} = \prod_{\varpi} (1 - \varpi^{-s}) \prod_{\varpi | n} \left(\frac{1 - \varpi^{(a+1)(1-s)}}{1 - \varpi^{1-s}}\right)$$

$$= \frac{1}{\zeta(s)} \sigma_{1-s}(n) = \frac{n^{1-s}}{\zeta(s)} \sigma_{s-1}(n)$$

where $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n. In particular, if we set $s = 2$, as $\zeta(2) = \pi^2/6$, we obtain the extremely illuminating formula

(12.7) $$\sigma(n) = 1/6\,(\pi^2 n)\left(1 + \frac{(-1)^n}{2^2} + \frac{2\cos 2n\pi/3}{3^2} + \frac{2\cos n\pi/2}{4^2} + \frac{2\cos 2n\pi/5 + 2\cos 4n\pi/5}{5^2} \cdots\cdots \right)$$

The series (12.6) is the <u>singular series</u> which is the key to the Hardy-Ramanujan work on the representation of numbers as sums of squares.

When s = 1, the series and product are no longer absolutely convergent, and the proof is not valid without further justification. But we have

(12.8) $$\sum \frac{c_q(n)}{q^s} = \sigma_{1-s}(n)\frac{}{\zeta(s)} = \sum_{d|n} d^{1-s} \sum \frac{\mu(n)}{n^s}$$

and it is known from the theory of primes that

(12.9) $$\sum \frac{\mu(n)}{n} = 0$$

Now the formal product of two Dirichlet series is convergent if one of them (here $\sum_{d|n} d^0$ which is a finite series) is absolutely convergent and the other is convergent. This yields

(12.10) $$\sum \frac{c_q(n)}{q} = 0$$

Also we may show that

(12.11) $$\sum \frac{c_q(n)}{q} \log q = d(n)$$

where $d(n) = \sigma_0(n) = \sum_{d|n} 1$

In the theory of the representation of numbers by sums of squares, the number 2 is exceptional in two ways, first according as the number to be represented is odd or even, second according to the power of 2 which divides the number of squares used.

We shall consider the theory of representing numbers by an even number of squares, 2s, and to simplify expressions involving certain roots of unity, we shall assume that $s \equiv 0$ (4). In particular we shall go through the details of the theory for representation by 8 squares or 24 squares.

First we consider the series

(12.12) $$S = 1^{-s} c_1(n) + 2^{-s} c_4(n) + 3^{-s} c_3(n) + 4^{-s} c_8(n) + \dots = \sum_{q=1}^{\infty} \alpha_q \, c_q(n)$$

where $\alpha_q = q^{-s}$ for $q \equiv 1, 3$ (4)

$= 0$ $\quad \equiv 2$ (4)

$= 2^s q^{-s}$ $\quad \equiv 0$ (4)

Our first problem is to sum this series. Suppose that

(12.14) $$q = 2^{\beta} \prod p^{\alpha} \qquad n = 2^b \prod p^a \quad , \text{ p's odd primes}$$

Then $$\alpha_q = 2^{\rho\sigma - \beta s} \prod p^{-\alpha s}$$

and

(12.15) $$S = \sum_{n=1}^{\infty} 2^{(\rho-\beta)s} \prod p^{-\alpha s} c_{2^\beta}(n) \prod c_{p^\alpha}(n) = \chi_2 \prod_p \chi_p$$

The evaluation of χ_p is the same as it was for (12.2)

(12.16) $$\chi_p = \sum_0^{\infty} p^{-\alpha s} c_{p^\alpha}(n) = (1-p^{-s})\left(\frac{1-p^{(\alpha+1)(1-s)}}{1-p^{1-s}}\right)$$

The evaluation of χ_2 is straightforward, but troublesome. Now

(12.17) $$\chi_2 = 1 + 0 + \sum_{\beta=2}^{\infty} 2^{(1-\beta)s} c_{2^\beta}(n)$$

where all $c_{2^\beta}(n)$ are 0 if n is odd.

Hence $\chi_2 = 1$ if n is odd and

(12.18) $$S = \prod_{p>2} (1-p^{-s})\left(\frac{1-p^{(a+1)(1-s)}}{1-p^{1-s}}\right) = \frac{\sigma_{1-s}(n)}{(1-2^{-s})\,\zeta(s)} = \frac{n^{1-s}\,\sigma_{s-1}(n)}{1^{-s}+3^{-s}+5^{-s}+\dots} \qquad \text{(n odd)}$$

When n is even as $2^b \| n$, we consider first b = 1, i.e. $n \equiv 2(4)$. Here $c_4(n) = -2$ and the rest of the c's are 0, and $\chi_2 = 1 - 2^{-s}$. For $b > 1$, we have

$$(12.19) \qquad = 1 + 2^{-s}.2 + 2^{-2s}.2^2 + \dots + 2^{(1-b)s}.2^{b-1} - 2^{-bs}.2^b$$
$$= 1 + 2^{1-s} + 2^{2-2s} + \dots + 2^{b(1-s)} - 2.2^{b(1-s)}$$

which incidentally gives the right values even for b = 1. Hence

$$(12.20) \qquad S = (1 + 2^{1-s} + \dots 2^{b(1-s)} - 2.2^{b(1-s)}) \prod_{p>2} (1-p^{-s}) \frac{(1-p^{(a+1)(1-s)})}{1-p^{1-s}}$$

$$\underline{\text{n even}} \qquad = (1 + 2^{1-s} + \dots + 2^{b(1-s)} - 2.2^{b(1-s)}) \frac{1}{1^{-s} + 3^{-s} + ..} \prod p^{a(1-s)} \prod \frac{p^{(a+1)(s-1)}-1}{p^{s-1}-1}$$

$$= \frac{1}{1^{-s} + 3^{-s} + 5^{-s} + \dots} \cdot \chi_2 \cdot \left(\frac{n}{2^b}\right)^{1-s} \sigma^{\circ}_{s-1}(n)$$

where $\sigma^{\circ}_{s-1}(n)$ is the sum of the $(s-1)^{st}$ powers of the <u>odd</u> divisors of n. But

$$(12.21) \qquad \sigma_{s-1}(n) = (1 + 2^{s-1} + \dots + 2^{b(s-1)}) \, \sigma^{\circ}_{s-1}(n)$$
$$= 2^{b(s-1)} (\chi_2 + 2.2^{b(1-s)}) \, \sigma^{\circ}_{s-1}(n)$$

whence

$$(12.22) \qquad 2^{b(s-1)} \chi_2 \, \sigma^{\circ}_{s-1}(n) = \sigma^{e}_{s-1}(n) - \sigma_{s-1}(n)$$

where $\sigma^{e}_{s-1}(n)$ is the sum of the (s-1) st powers of the <u>even</u> divisors of n. Substituting in (12.20) we obtain finally

$$(12.23) \qquad S = \frac{n^{1-s}(\sigma^{e}_{s-1}(n) - \sigma^{\circ}_{s-1}(n))}{1^{-s} + 3^{-s} + 5^{-s} + \dots}$$

<u>n even</u>

We may combine (12.18) and (12.23) into

$$(12.24) \qquad (1^{-s} + 3^{-s} + 5^{-s} + \dots)\, n^{s-1} S = \sigma^{*}_{s-1}(n) = \begin{cases} \sigma_{s-1}(n) & \underline{\text{n odd}} \\ \sigma^{e}_{s-1}(n) - \sigma^{\circ}_{s-1}(n) & \underline{\text{n even}} \end{cases}$$

and we have thus evaluated the series.

Before proceeding further it is interesting to note an identity involving divisor functions conjectured by Ramanujan and proved by B. M. Wilson.

The identity of Ramanujan's which Ingham used to prove that $\zeta(1+ti) \neq 0$ is

$$(12.25) \qquad S = \sum n^{-s} \sigma_a(n) \sigma_b(n) = \frac{\zeta(s)\, \zeta(s-a)\, \zeta(s-b)\, \zeta(s-a-b)}{\zeta(2s-a-b)}$$

If we write

$$S = \prod_{\varpi} \chi_{\varpi}$$

we have

$$\chi_{\varpi} = 1 + \varpi^{-s} \sigma_a(\varpi)\, \sigma_b(\varpi)$$
$$= 1 + \varpi^{-s}(1+\varpi^a)(1+\varpi^b) + \varpi^{-2s}(1+\varpi^a+\varpi^{2a})(1+\varpi^b+\varpi^{2b}) + \dots$$
$$= 1 + \varpi^{-s} \frac{(\varpi^{2a}-1)}{(\varpi^a-1)} \frac{(\varpi^{2b}-1)}{(\varpi^b-1)} + \varpi^{-2s} \frac{(\varpi^{3a}-1)}{(\varpi^a-1)} \frac{(\varpi^{3b}-1)}{(\varpi^b-1)} + \dots$$
$$= \frac{1}{(\varpi^a-1)(\varpi^b-1)} \left[\varpi^{a+b} - \varpi^a - \varpi^b + 1 + \varpi^{-s}(\varpi^{2a+2b} - \varpi^{2a} - \varpi^{2b} + 1) + \dots \right]$$
$$= \frac{1}{(\varpi^a-1)(\varpi^b-1)} \left[\frac{\varpi^{a+b}}{1-\varpi^{a+b-s}} - \frac{\varpi^a}{(1-\varpi^{a-s})} - \frac{\varpi^b}{1-\varpi^{b-s}} + \frac{1}{1-\varpi^{-s}} \right]$$
$$= \frac{1 - \varpi^{a+b-2s}}{(1-\varpi^{-s})(1-\varpi^{a-s})(1-\varpi^{b-s})(1-\varpi^{a+b-s})}$$

Now remembering that

$$\zeta(s) = \prod \chi_{\varpi}$$

where

$$\chi_{\varpi} = \frac{1}{1 - \varpi^{-s}}$$

we see at once that this expression has the value

$$\frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-b)}$$

as was to be shown. Note that if we put $a = b = 0$, we obtain

$$\sum n^{-s}\,(d(n))^2 = \frac{\zeta^4(s)}{\zeta(2s)}$$

From the identity (12.25), A. E. Ingham has deduced an ingenious proof that $\zeta(s)$ has no zeros on the line $\sigma = 1$. For suppose that

(12.26) $$\zeta(1 + i\alpha) = 0$$

(whence also $\zeta(1 - i\alpha) = 0$).

Then in (12.25) set $a = i\alpha$, $b = -i\alpha$, yielding

(12.27) $$\frac{\zeta^2(s)\,\zeta(s-i\alpha)\,\zeta(s+i\alpha)}{\zeta(2s)} = \sum_{n=1}^{\infty} n^{-s}\,\sigma_{i\alpha}(n)\,\sigma_{-i\alpha}(n)$$

Now $\sigma_{i\alpha}(n)\,\sigma_{-i\alpha}(n)$ must be real and positive for every n. By a theorem of Landau's such a series must have a singularity on its line of convergence. On the left hand side of (12.27) the denominator, $\zeta(2s)$, has no zeros or poles for $\sigma > \frac{1}{2}$. $\zeta(s)$ has a simple pole for $s = 1$, but this in $\zeta^2(s)$ is canceled by the zeros of $\zeta(s - i\alpha)$ and $\zeta(s + i\alpha)$ for $s = 1$. $\zeta(s - i\alpha)$ has a simple pole for $s = 1 + i\alpha$ which is canceled by the zero of $\zeta(s)$ for this value. Similarly the pole of $\zeta(s + i\alpha)$ at $s = 1 - i\alpha$ is canceled by the zero of $\zeta(s)$. Hence the numerator is regular for all finite s. Consequently, the series on the right of (12.27) must converge for all $\sigma > \frac{1}{2}$. Now in this series the constant term is 1 and all other coefficients are positive. Hence it is always numerically greater than 1 for any real s greater than $\frac{1}{2}$. That is, since $\zeta(s + i\alpha)$ and $\zeta(s - i\alpha)$ are conjugate for real s

(12.28) $$\frac{\zeta^2(\sigma)\,|\zeta(\sigma + i\alpha)|^2}{\zeta(2\sigma)} > 1$$

or

(12.29) $$\zeta^2(\sigma)\,|\zeta(\sigma + i\alpha)|^2 > \zeta(2\sigma)$$

But as $s = 1$ is the only finite singularity of $\zeta(s)$, $\zeta(\frac{1}{2})$ and $\zeta(\frac{1}{2} + i\alpha)$ are finite and (12.29) implies that as $\sigma \to \frac{1}{2}$ through real values that $\zeta(2\sigma)$ remains bounded. But $\zeta(s)$ has a simple pole for $s = 1$, and this cannot be the case. Hence our assumption (12.26) must be false, and we conclude that $\zeta(s)$ has no zeros on the line $\sigma = 1$.

<u>The singular series in the problem of 2s squares.</u>

The series (12.12) and its evaluation (12.24) play an important role in the theory of representing numbers by sums of squares. Analytically we write

(12.30) $$(1 + 2x + 2x^4 + \dots)^{2s} = 1 + \sum r_{2s}(n)\,x^n$$

as a function which enumerates the number $r_{2s}(n)$ of representations of n as a sum of 2s squares. Note that in these <u>representations</u> we count both order and sign. Thus

(12.31) $$5 = (\pm 1)^2 + (\pm 2)^2 = (\pm 2)^2 + (\pm 1)^2$$

has eight representations as a sum of two squares. It has been known since the time of Jacobi that

(12.32) $$r_{2s}(n) = \delta_{2s}(n) \qquad \text{for } 2s = 2, 4, 6, 8$$

where $\delta_{2s}(n)$ is a divisor function. Thus

(12.33) $$r_8(n) = 16 \;\sigma_3^*(n)$$

where σ^* has the same meaning as in (12.24). In general it is true that

(12.34) $$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n)$$

where $\delta_{2s}(n)$ is a divisor function and $e_{2s}(n)$ is a function of order considerably lower than that of $\delta_{2s}(n)$. (In Ramanujan's papers the notation is slightly changed so that his δ and e are half of those used here.) We shall prove that when $s \equiv 0 \ (4)$ we have

(12.35) $$\delta_{2s}(n) = \frac{\pi^s}{\Gamma(s)\,(1^{-s} + 3^{-s} + 5^{-s} + \ldots)}\,\sigma^*_{s-1}(n)$$

When $s \not\equiv 0 \ (4)$ there are slight formal variations depending on the residue of s (mod 4). We note that the first factor in s may be calculated from the values of the Bernouilli numbers. Moreover it is true that

(12.36) $$e_{2s}(n) = 0 \qquad 2s = 2, 4, 6, 8$$

and we shall prove this for 2s = 8. As an example of higher values of 2s we shall consider 2s = 24 and prove

(12.37) $$\frac{691}{128}\, e_{24}(n) = (-1)^n\, 259\,\tau(n) - 512\;\tau(\tfrac{1}{2}n)$$

where

(12.38) $$\sum \tau(n)\, x^n = x\left\{(1-x)(1-x^2)\ \ldots\ \right\}^{24}$$

In (12.37) we take $\tau(\frac{1}{2}n)$ to be zero if n is odd. Now $\delta_{24}(n)$ is of order n^{11} and $\tau(n)$ is (as we shall show later) of order not exceeding n^7.

The series on the left of (12.30) is of course one of the familiar ϑ-functions. We shall draw freely upon the known properties of ϑ-functions as given in Tannéry and Molk, and shall follow their notation. We write

(12.39) $$f(x) = (1 + 2x + 2x^4 + \ldots\,)^{2s} = \vartheta_3^{2s}(0,\tau) = \vartheta^{2s}$$

where $x = e^{\pi i \tau}$. This function has the unit circle as a natural boundary, but although every point of the unit circle is a singularity, the 'rational points'

(12.40) $$x_{p,\,q} = e^{2p\pi i/q}$$

stand out among them. We study the behaviour of f(x) as $x \to x_{p,\,q}$ along a radius, $x = r\, e^{2p\pi i/q}$ as $r \uparrow 1$. Now

(12.41) $$\begin{aligned} \vartheta &= 1 + 2\sum_{n=1}^{\infty} r^{n^2}\, e^{2n^2 p\pi i/q} \\ &= 1 + 2\sum_{j=0}^{q-1}\sum_{\ell=0}^{\infty} r^{(2q + j)^2}\, e^{2(\ell q + j)^2\, p2\pi i/q} \\ &= 1 + 2\sum_{j=0}^{q-1} e^{2j^2 p\pi i/q} \sum_{\ell=0}^{\infty} r^{(\ell q + j)^2} \end{aligned}$$

Here

(12.42) $$\begin{aligned} \sum_{\ell=0}^{\infty} r^{(\ell q + j)^2} &\sim \frac{\sqrt{\pi}}{2q}\,(\log 1/r)^{-1/2} \\ &\sim \frac{\sqrt{\pi}}{2q}\,(1 - r)^{-1/2} \end{aligned}$$

as $r \uparrow 1$. Hence

(12.43) $$\vartheta \sim \sqrt{\pi}\,\frac{S_{p,\,q}}{q}\,(\log 1/r)^2 \qquad \text{as } r \uparrow 1$$

where

$$(12.44)\qquad S_{p,q} = \sum_{j=0}^{q-1} e^{2j^2 p\pi i/q}$$

If $S_{p,q} = 0$, we interpret (12.43) as $\vartheta = o\,(\log 1/r)^{-1/2}$ as $r \uparrow 1$. We now make a heuristic construction of a function which shall mimic ϑ^{2s} near <u>all the points</u> $x_{p,q}$, being guided by (12.43). We write

$$(12.45)\qquad f_{p,q}(x) = \frac{\pi^s}{\Gamma(s)}\left(\frac{S_{p,q}}{q}\right)^{2s} F_s(x\, e^{-2p\pi i/q})$$

where

$$(12.46)\qquad F_s(x) = \sum_{1}^{\infty} n^{s-1} x^n.$$

In fact

$$(12.47)\qquad F_s(x) - \Gamma(s)\left(\log 1/x\right)^{-s}$$

is regular near $x = 1$. We now construct

$$(12.48)\qquad \Theta_{2s}(x) = 1 + \sum f_{p,q}(x) = \sum \rho_{2s}(n)\, x^n$$

summing over all q and all $p \leqq q$ and prime to q. Here

$$(12.49)\qquad \rho_{2s}(n) = \sum_{p,q} c_{p,q,n}$$

where

$$(12.50)\qquad c_{p,q,n} = \frac{\pi^s n^{s-1}}{\Gamma(s)}\left(\frac{S_{p,q}}{q}\right)^{2s} e^{-2np\pi i/q}$$

whence

$$(12.51)\qquad \rho_{2s}(n) = \frac{\pi^s n^{s-1}}{\Gamma(s)} \sum_{1}^{\infty} A_q$$

where

$$(12.52)\qquad A_q = q^{-2s} \sum_{p} (S_{p,q})^{2s} e^{-2np\pi i/q}.$$

We call $\rho_{2s}(n)$ the <u>singular series</u>. We must now do two things: (1) sum the singular series, and (2) prove that actually $\Theta_{2s} = \vartheta^{2s}$ whence $\rho_{2s}(n) = r_{2s}(n)$ as for $2s = 8$ or that $\Theta_{2s} = \vartheta^{2s} + R_{2s}$ where $R_{2s}(x) = \sum \varrho_{2s}(n)\, x^n$ is another function of x to be determined. The singular series for $\rho_{2s}(n)$ turns out to be, except for a simple factor, the series (12.12).

For the $S_{p,q}$ of (12.52) we use the known results on Gaussian sums

$$(12.53)\qquad S_{p,qq'} = S_{p,q} \cdot S_{p,q'} \quad \text{where } (q, q') = 1.$$ For q a power of 2, we have

$$(12.54)\qquad S_1 = 1,\quad S_2 = 0,\quad S_{p,2^{2\mu}} = 2^{\mu}(1 + i^p)$$

$$S_{p,2^{2\mu+1}} = 2^{\mu+1} e^{1/4\, p\pi i}$$

For q a power of an odd prime ϖ

$$(12.55)\qquad S_{\varpi} = \left(\frac{p}{\varpi}\right) i^{\frac{1}{4}(\varpi-1)^2}\sqrt{\varpi}$$

$$S_{\varpi^{2\mu}} = \varpi^{\mu}$$

$$S_{\varpi^{2\mu+1}} = \varpi^{\mu} S_{\varpi}$$

If we observe that $(1 \pm i)^{2s} = 2^s$ when $s \equiv 0\ (4)$, and that all roots of unity involved become one when $s \equiv 0\ (4)$, we have

$$(12.56)\qquad S_{p,q}^{2s} = \begin{cases} q^s & \text{if } q \equiv 1, 3 \quad (4) \\ 0 & q \equiv 2 \quad (4) \\ 2^s q^s & q \equiv 0 \quad (4) \end{cases}$$

Making these substitutions in (12.52)

$$(12.57)\qquad A_q = q^{-s}\eta_q \sum_{(p,q)=1} e^{-2np\pi i/q}$$

where $\eta_q = 1$ if $q \equiv 1, 3 \ (4)$
$\quad = 0 \quad q \equiv 2 \quad (4)$
$\quad = 2^s \quad q \equiv 0 \quad (4)$

Hence $A_q = q^{-s}\eta_q\, c_q(n)$ and

(12.58) $$\rho_{2s}(n) = \frac{\pi^s n^{s-1}}{\Gamma(s)}\left(1^{-s} c_1(n) + 2^{-s} c_4(n) + 3^{-s} c_3(n) + \dots \right)$$

so that when $s \equiv 0 \ (4)$ $\rho_{2s}(n)$ is the series of (12.12) apart from the factor $\dfrac{\pi^s n^{s-1}}{\Gamma(s)}$. When $s \not\equiv 0 \ (4)$, we evidently must make some more slight changes because of the factors appearing in (12.54) and (12.55). Combining (12.58) and (12.24) we have

(12.59) $$\rho_{2s}(n) = \frac{\pi^s}{\Gamma(s)\,(1^{-s} + 3^{-s} + \dots)}\,\sigma^*_{s-1}(n)$$

the desired evaluation of the singular series.

13. The elliptic modular functions.

In the preceding section we outlined the method for constructing a function Θ_{2s} which should mimic ϑ^{2s} as closely as possible near the singularities of ϑ^{2s} and Θ_{2s} led us to the singular series $\rho_{2s}(n)$ of (12.58) and evaluated in (12.59). We wish to be able to prove that $\rho_{2s}(n)$ is either equal or nearly equal to $r_{2s}(n)$. So far our procedure has been in a large measure formal.

We now turn to the main analytic problem, that of determining to what extent Θ_{2s} approximates ϑ^{2s}. This leads us into the domain of the elliptic modular functions. We shall give here a brief sketch of the fundamental ideas of this subject so far as they are required here.

The homogeneous modular group Γ, consists of all substitutions of the form

(13.1) $$\omega_1' = a\,\omega_1 + b\,\omega_2$$
$$\omega_2' = c\,\omega_1 + d\,\omega_2$$

where a, b, c, d are rational integers and $ad - bc = 1$. If ω_1 and ω_2 are half-periods of an elliptic function, we write

(13.2) $$\tau = \frac{\omega_1}{\omega_2} \qquad \mathcal{I}(\tau) > 0$$

The relation $\mathcal{I}(\tau) > 0$ will be preserved by transformations of the modular group and hence we may restrict our attention to the part of the complex plane above the real axis.

The best short account of the theory of modular functions is contained in two papers by Hurwitz in volumes 18 and 52 of the Mathematische Annalen. For a detailed account, there is the book of Klein and Fricke.

Using (13.2) we may consider the group in a non-homogeneous form. In this form the group is generated by the two transformations

(13.3) $$\tau' = \tau + 1 \qquad \text{and} \qquad \tau' = -1/\tau \quad .$$

Every point in the upper half plane may be transformed by a substitution of the modular group into one and only one point within the fundamental domain D defined by

(13.4) $$-1/2 < \mathcal{R}(\tau) \leq +\tfrac{1}{2},\ \mathcal{I}(\tau) > 0,\ |\tau| \geq 1,\ \mathcal{R}(\tau) \geq 0 \text{ on } |\tau| = 1$$

This is a curvilinear triangle of angles $\frac{\pi}{3}$, $\frac{\pi}{3}$, and 0. Writing $x = e^{\pi i \tau}$ as before, we define

(13.5) $$\Delta(\omega_1, \omega_2) = \left(\frac{2\pi}{\omega_1}\right)^{12} x^2 \prod (1 - x^{2n})^{24}$$
$$g_2 = 1/12 \left(\frac{\pi}{\omega_2}\right)^4 \left\{1 + 240\left(\frac{1^3 x^2}{1-x^2} + \frac{2^3 x^4}{1-x^4} + \frac{3^3 x^6}{1-x^6} + \dots\right)\right\}$$
$$g_3 = 1/216 \left(\frac{\pi}{\omega_2}\right)^6 \left\{1 - 504\left(\frac{1^5 x^2}{1-x^2} + \frac{2^5 x^4}{1-x^4} + \frac{3^5 x^6}{1-x^6} + \dots\right)\right\}$$

(when $\Delta = g_2^{\,3} - 27\, g_3^{\,2}$)

$$J(\tau) = \frac{g_2^3}{\Delta}$$

The most fundamental relation of all is

(13.6) $$J(\tau') = J(\tau)$$

where τ' is any transform of τ by a substitution of the modular group. Moreover $J(\tau)$ takes on every value exactly once within D, and hence defines a conformal mapping of D on to the whole plane. Thus $J(\tau)$ plays the role for modular functions which e^z plays for simply periodic functions and $\wp(z)$ for doubly periodic functions, and z itself for one-valued functions in general. Every function invariant under the modular group is a single valued function of $J(\tau)$. We shall use for our investigation of Θ_{2s} the following principle:

<u>If a modular invariant is bounded in D it is a constant.</u> This is true since a modular invariant bounded in D must be bounded over the entire J plane.

We make use, not of the entire modular group, but of a subgroup Γ_3 of index 3, namely the group of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying

(13.7) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (\text{mod } 2).$$

This group is generated by

(13.8) $$\tau' = \tau + 2 \quad \text{and} \quad \tau' = -1/\tau$$

For Γ_3 the fundamental domain is D_3 defined by

(13.9) $$-1 < \Re(\tau) \leq +1, \quad \Im(\tau) > 0, \quad |\tau| > 1$$

and there is an invariant J_3 fundamental for Γ_3. Now an invariant of Γ_3 is a one-valued function of J_3, but in general a three-valued function of J. As before

<u>A function invariant under Γ_3, which is bounded in D_3 must be a constant.</u> We shall prove

(13.10)

(1) $\dfrac{\Theta_8(x)}{\vartheta^8(x)}$ is invariant under Γ_3

(2) $\dfrac{\Theta_8(x)}{\vartheta^8(x)}$ is bounded in D_3.

Knowing these facts, it is easy to calculate that $\dfrac{\Theta_8(x)}{\vartheta^8(x)} = 1$. More generally, we shall prove

(1) $\dfrac{\Theta_{2s}}{\vartheta^{2s}}$ is invariant under Γ_3 and

(2) $\dfrac{\Theta_{2s}}{\vartheta^{2s}} + \sum \dfrac{I_i}{\vartheta^{2s}}$ is bounded in D_3 where $\dfrac{I_i}{\vartheta^{2s}}$ are appropriate invariants under Γ_3.

We begin by proving the necessary invariance. From table 42, volume 2 of Tannéry and Molk, we have for $s \equiv 0 \ (4)$

(13.11) $$\vartheta^{2s}(\tau + 1) = \vartheta^{2s}(\tau)$$
$$\vartheta^{2s}(-1/\tau) = \tau^s \vartheta^{2s}(\tau)$$

Hence ϑ^{2s} is invariant under Γ and à fortiori under Γ_3. Now

(13.12) $$\Theta_{2s} = 1 + \sum_{p,q} f_{p,q}(x)$$

where

(13.13) $$f_{p,q}(x) = \frac{\pi^s}{\Gamma(s)} \frac{\eta_q}{q^s} F_s(xe^{-2p\pi i/q})$$

and if $x = e^{-y}$

(13.14) $$F_s(x) = \sum_1^{\infty} n^{s-1} e^{-ny} = \left(\frac{d}{dy}\right)^{s-2} \sum n e^{-ny}$$

$$=\left(\frac{d}{dy}\right)^{s-2}\left(1/4 \operatorname{cosech}^2 y/2\right)$$

$$=\left(\frac{d}{dy}\right)^{s-2}\sum_{n} \frac{1}{(y+2n\pi i)^2} \quad = \quad \Gamma(s)\sum \frac{1}{(y+2n\pi i)^s}$$

Here since $y = -\pi i \tau$

(13.15) $$F_s(x) = \frac{\Gamma(s)}{\pi^s}\sum_{n} \frac{1}{(2n-\tau)^s}$$

and

(13.16) $$F_s(x\, e^{-2p\pi i/q}) = \frac{\Gamma(s)}{\pi^s}\sum_{n} \frac{1}{(2n - 2 + 2p/q)^s}$$

whence

(13.17) $$f_{p,\,q}(x) = \eta_q \sum_{n} \frac{1}{(2\,\overline{nq+p} - \tau q)^s}$$

and

(13.18) $$\Theta_{25} = 1 + \sum_{p,q,n} \frac{\eta_q}{(2\,\overline{nq+p} - q\tau)^s}$$

Here matters are somewhat complicated by the range over which the indices of summation run.

$$q = 1, 2, 3, \ldots$$
$$0 < p \leqq q \qquad (p,q) = 1$$
$$-\infty < n < +\infty$$

Here we may replace $nq + p$ by p if we replace $0 < p \leqq q$ $(p,q) = 1$ by the condition $(p,q) = 1$.
Hence, remembering the values of η_q from (12.57)

(13.19) $$\begin{aligned}\Theta_{25} &= 1 + \sum_{q=1,3,\ldots} \frac{1}{(2p - q\tau)^s} + \sum_{q=4,8,\ldots} \frac{2^s}{(2p - q\tau)^s}\\ &= 1 + \sum_{q=1,3,\ldots} \frac{1}{(2p - q\tau)^s} + \sum_{q=2,4,\ldots} \frac{1}{(p - q\tau)^s}\\ &= 1 + \sum \frac{1}{(p - q\tau)^s}\end{aligned}$$

where in the last sum we require only $(p,q) = 1$. Now multiply by $\eta(s) = 1^{-s} + 3^{-s} + 5^{-s} + \ldots$

(13.20) $$\eta(s)\,\Theta_{25} = \eta(s) + \sum_{q=1,2,\ldots} \frac{1}{(p - q\tau)^s}$$

where now p runs over all values of opposite parity to q, and $q = 0$ is excluded. We may write (13.20) in two ways:

(13.21) $$\eta(s)\,\Theta_{25}(x) = \sum_{p,q \text{ opposite parity}} \frac{1}{(p - q\tau)^s}$$

or

(13.22) $$\eta(s)\,\Theta_{25}(x) = \eta(s) + \tfrac{1}{2}\sum_{q\neq 0}\sum_{p} \frac{1}{(p - q\tau)^s}$$
$$p,q \text{ opposite parity}$$

From (13.21) Θ_{25} is invariant under the substitution $\tau \to \tau + 2$. (note that $\eta(s)$ is independent of τ).
From (13.22) replacing τ by $-1/\tau$ we obtain

(13.23) $$\begin{aligned}\chi(-1/\tau) &= \eta(s) + \tfrac{1}{2}\tau^s \sum{}' \sum \frac{1}{(p\tau + q)^s}\\ &= \eta(s)\tau^s + \tfrac{1}{2}\tau^s \sum{}'' \frac{1}{(p\tau + q)^s} \quad = \tau^s \chi(\tau)\end{aligned}$$

where $\sum''$ means omitting $p = 0$. Hence comparing with (13.1) $\dfrac{\Theta_{25}}{\vartheta^{25}}$ is invariant in Γ_3.

The invariance of the quotient $\frac{\Theta_{2s}}{\vartheta^{2s}}$ under Γ_3 has now been proved in general, but in general this quotient is not bounded. But we sha ll prove that $\frac{\Theta_8}{\vartheta^8}$ is bounded. We may prove that ϑ (x) has no zeros for $|x| < 1$, i.e. for $\mathcal{J}(\tau) > 0$ by using the product expansion (8.16) with $q = x$, $z = i$. Moreover we need not worry about finite values of $|x| < 1$ and τ, connected by $x = e^{\pi i \tau}$ because of the convergence of the series for ϑ_8 and Θ_8. The only infinite value for τ in D_3 is $i\infty$ and to this corresponds $x = 0$ for which the series both converge to unity. However in D_3 there are two values for which $\vartheta(\tau) = 0$ and $|x| = 1$, namely $\tau = -1, +1$, of which we need consider only +1, since we exclude the left boundary of D_3.

Write

(13.24) $$\tau = 1 - \frac{1}{T} \qquad X = e^{\pi i T}$$

Hence as $\tau \to 1$ from the interior of D_3, $T \to +i\infty$, and $X \to 0$. Now

(13.25) $$\begin{aligned}\vartheta_3^8(0, 1 - \tfrac{1}{T}) &= T^4(\vartheta_2(0, T))^8 \\ &= T^4(2X^{\frac{1}{4}} + 2X^{\frac{9}{4}} + \dots)^8 \\ &= 256\, T^4(X^2 + \dots)\end{aligned}$$

Hence if we can show

(13.26) $$\Theta_8(1 - \tfrac{1}{T}) = A\, T^4 X^2 + \dots$$

then we shall have proved that $\frac{\Theta_8}{\vartheta^8}$ is bounded for $\tau = +1$, completing the proof of its boundedness. We have

(13.27) $$\eta(8)\Theta_8 = \eta(8) + \tfrac{1}{2}\sum{}' \sum \frac{1}{(p - q\tau)^4} = \chi(\tau)$$

where

(13.28) $$\eta(8) = \frac{1}{1^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{96}$$

Hence

(13.29) $$\begin{aligned}\chi(1 - 1/T) &= \frac{\pi^4}{96} + \tfrac{1}{2}T^4 \sum{}' \sum \frac{1}{(\overline{p-q}\,T + q)^4} \\ &= \frac{\pi^4}{96} + \tfrac{1}{2}T^4 \sum{}' \sum \frac{1}{(Q + PT)^4}\end{aligned}$$

where we require $Q \neq 0$ and P odd. We incorporate the series $\eta(8)$ into the sum, replacing the terms for $Q = 0$. Thus

(13.30) $$\chi(1 - 1/T) = \tfrac{1}{2}T^4 \sum\sum \frac{1}{(Q + PT)^4}$$

summed over all Q and all odd P. if we put $PT = a$, and $\zeta = e^{\pi i a} = X^P$, (13.30) becomes

(13.31) $$\begin{aligned}\frac{2\chi(1 - 1/T)}{T^4} &= \sum_a \sum_Q \frac{1}{(Q+a)^4} \\ &= \sum \tfrac{1}{6}\left(\frac{d}{da}\right)^2 \sum \frac{1}{(Q+a)^2} \\ &= \sum_a 1/6 \left(\frac{d}{da}\right)^2 \pi^2 \operatorname{cosec}^2 a\pi \\ &= \sum_a -1/6 \left(\frac{d}{da}\right)^2 \frac{4\pi^2 e^{2a\pi i}}{(1 - e^{2a\pi i})} \\ &= \sum_a -2\pi^2/3 \left(\frac{d}{da}\right)^2 e^{2a\pi i}(1 + 2e^{2a\pi i} + 3e^{4a\pi i} + \dots) \\ &= 8\pi^4/3 \sum_a (1^3 e^{2a\pi i} + 2^3 e^{4a\pi i} + 3^3 e^{6a\pi i} + \dots) \\ &= 8\pi^4/3 \sum_P (1^3 X^{2P} + 2^3 X^{4P} + \dots)\end{aligned}$$

where we sum over P odd. Thus the least exponent of X is 2. Hence

(13.32) $$\eta(8)\,\Theta_8 = \chi(1 - 1/T) = cT^4(X^2 + \dots)$$

and we have proved that $\frac{\Theta_8}{\vartheta^8}$ is bounded in D_3. Hence $\frac{\Theta_8}{\vartheta^8}$ is a constant and we easily calculate that this constant is one.

We have already made use of a few formulae for ϑ-functions. We shall have occasion to use several more, and we give here the essential formulae, as copied from Tannéry and Molk.

(13.33)
$$q = e^{\pi i \tau}$$
$$h(\tau) = q^{1/12}(1-q^2)(1-q^4)..$$
$$h(\tau+1) = \sqrt[6]{i}\ \ h(\tau)$$
$$h(-\tfrac{1}{\tau}) = \frac{\sqrt{\tau}}{\sqrt{i}}\, h(\tau)$$
$$h^2\left(\frac{\tau+1}{2}\right) = \sqrt[6]{i}\ \ h(\tau)\,\vartheta_3(0,\tau)$$
$$\vartheta_2(0,\tau) = 2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + \ldots$$
$$\vartheta_3(0,\tau) = 1 + 2q + 2q^4 + ..$$
$$\vartheta_4(0,\tau) = 1 - 2q + 2q^4 + \ldots$$
$$\vartheta_2(0,\tau+1) = \sqrt{i}\ \vartheta_2(0,\tau)$$
$$\vartheta_3(0,\tau+1) = \vartheta_4(0,\tau)$$
$$\vartheta_4(0,\tau+1) = \vartheta_3(0,\tau)$$
$$\vartheta_2(0,-\tfrac{1}{\tau}) = \frac{\sqrt{\tau}}{\sqrt{i}}\vartheta_4(0,\tau)$$
$$\vartheta_3(0,-\tfrac{1}{\tau}) = \frac{\sqrt{\tau}}{\sqrt{i}}\vartheta_3(0,\tau)$$
$$\vartheta_4(0,-\tfrac{1}{\tau}) = \frac{\sqrt{\tau}}{\sqrt{i}}\ \vartheta_2(0,\tau)$$

All these are formulae for linear transformation, except those for $h^2\left(\frac{\tau+1}{2}\right)$ (quadratic transformation.)

We now consider the case $2s = 24$. The proof of the invariance of $\frac{\Theta_{24}}{\vartheta^{24}}$ under Γ_3 is included in the general proof of the invariance of $\frac{\Theta_{2s}}{\vartheta^{2s}}$. We prove

(13.34)
$$\vartheta^{24} = \Theta_{24} - \frac{33152}{691}f(-x) - \frac{65536}{691}f(x^2)\ .$$

where

(13.35)
$$f(x) = x\ \left[(1-x)(1-x^2)\ldots.\right]^{24}$$
$$= \sum_{1}^{\infty}\tau(n)\ x^n$$

which is paraphrased into

(13.36)
$$r_{24}(n) = \delta_{24}(n) + e_{24}(n)$$

where

(13.37)
$$e_{24}(n) = \frac{128}{691}\left\{(-1)^{n-1}\ 259\ \tau(n) - 512\ \tau(n/2)\right\}$$

taking $\tau(n/2) = 0$ when n is odd, and

(13.38)
$$\delta_{24}(n) = \frac{16}{691}\ \sigma^*_{11}(n)$$

Here $\delta_{24}(n)$ is of the exact order n^{11}. There is a fairly simple proof of Ramanujan's that

(13.39)
$$\tau(n) = O(n^7).$$

A more complicated proof of Hardy's shows that

(13.40)
$$\tau(n) = O(n^6)$$

and it seems highly probable that

(13.41)
$$\tau(n) = O(n^{11/2+\varepsilon}).$$

In order to prove (13.34) we shall show that $\frac{f(-x)}{\vartheta^{24}}$ and $\frac{f(x^2)}{\vartheta^{24}}$ are invariant under Γ_3, and that $\dfrac{\Theta_{24} + \alpha f(-x) + \beta f(x^2)}{\vartheta^{24}}$ is bounded in D_3 if α and β are appropriately chosen constants. When this is done, the calculation of α and β is quite elementary though somewhat tedious. Comparing (13.35) with (13.33) we see that

(13.42) $$f(x^2) = h^{24}(\tau)$$

and again from (13.33)

(13.43) $$h^{24}(\tau + 1) = h^{24}(\tau)$$
$$h^{24}\left(-\tfrac{1}{\tau}\right) = \tau^{12}\, h^{24}(\tau)$$

and as $\vartheta = \vartheta_3(0,\tau)$ it follows that $\dfrac{f(x^2)}{\vartheta^{24}}$ is invariant not only under Γ_3 but under Γ.

Again from (13.33) and (13.35)

(13.44) $$f(-x) = h^{24}\left(\tfrac{\tau+1}{2}\right) = -\,(h(\tau))^{12}\,(\vartheta_3(0,\tau))^{12}$$

whence $\dfrac{f(-x)}{\vartheta^{24}}$ is invariant under Γ_3. Having found appropriate functions, $f(x^2)$ and $f(-x)$, whose quotient with ϑ^{24} is invariant under Γ_3, we have overcome the principal difficulty in evaluating ϑ^{24} and now proceed to the more direct problem of proving that $\dfrac{\Theta_{24} + \alpha f(-x) + \beta f(x^2)}{\vartheta^{24}}$ is bounded in D_3 when α and β are appropriately chosen. As was the case previously, this function is immediately seen to be bounded everywhere in D_3 except possibly near the points $\tau = -1, +1$ of which we need consider only $\tau = +1$. Putting $\tau = 1 - 1/T$ as before, we let $T \to i\infty$. Using the appropriate formulae from (13.33):

(13.45) $$\begin{aligned} f(x^2) &= h^{24}(\tau) \\ &= h^{24}(1 - 1/T) = h^{24}(-1/T) \\ &= T^{12}\, h^{24}(T) = T^{12}\, (X^2 \{(1 - X^2)^{24} \cdots\}) \end{aligned}$$

(13.46) $$\begin{aligned} f(-x) &= -h^{12}(\tau)\,\vartheta_3^{12}\left(0, 1 - \tfrac{1}{T}\right) \\ &= -h^{12}(1 - 1/T)\,\vartheta_3^{12}\left(0, 1 - \tfrac{1}{T}\right) \\ &= h^{12}(-1/T)\,\vartheta_4^{12}\left(0, -\tfrac{1}{T}\right) \\ &= \left(\tfrac{\sqrt{T}}{\sqrt{i}}\right)^{12}\left(\tfrac{\sqrt{T}}{\sqrt{i}}\right)^{12} \ell^{12}(T)\,\vartheta_2^{12}(0,T) \\ &= T^{12}\, X\, (1 - X^2)^{12} \cdots\cdots (2X^{\frac{1}{4}} + 2X^{\frac{9}{4}} + \cdots)^{12} \end{aligned}$$

Now we argue on Θ_{24} as we did on Θ_8 and we reach

(13.47) $$\begin{aligned} \sum \frac{1}{(Q+a)^{12}} &= \frac{1}{11!}\left(\frac{d}{da}\right)^{10} \sum \frac{1}{(Q+a)^2} \\ &= \frac{(2\pi)^{12}}{11!}\,(1^{11} X^{2P} + 2^{11} X^{2P} + \cdots) \end{aligned}$$

Hence

(13.48) $$\Theta_{24} = \frac{(2\pi)^{12} T^{12}}{\eta(12)\; 11!}\,(X^2 + 2^{11} X^4 + \cdots)$$

where

(13.49) $$\eta(12) = \frac{1}{1^{12}} + \frac{1}{3^{12}} + \cdots$$

Also

(13.50) $$\begin{aligned} \vartheta^{24}\left(0, 1 - \tfrac{1}{T}\right) &= T^{12}(2X^{\frac{1}{4}} + 2X^{\frac{9}{4}} + \cdots)^{24} \\ &= T^{12} X^6 (A + B X^2 + \cdots) \end{aligned}$$

Now the series for Θ_{24} begins with X^2, that for $f(x^2)$ with X^2, that for $f(-x)$ with X^4, and that for ϑ^{24} with X^6. Hence we may choose α and β so that in $\Theta_{24} + \alpha f(x^2) + \beta f(-x)$ the terms in X^2 and X^4 drop out and hence as a series beginning with X^6 its quotient by ϑ^{24} will be bounded near $X = 0$. This completes the proof and evaluation of constants leads us to the desired equation (13.34).

Ramanujan stated ["Certain Arithmetical Functions", Transactions of the Cambridge Philosophical Society, Vol. 22, No. 9 (1916) pp. 159 - 184] and Mordell proved correct [Quarterly Journal of Mathematics, Vol. 48 (1917) pp. 93 - 104] a method of writing down invariants like the $f(x^2)$ and $f(-x)$ for Θ_{24} which will enable us to evaluate any ϑ^{2s} though it will not necessarily give the happiest forms.

It was known to Liouville and proved by Glaisher that

(13.51) $$r_{10}(n) = \delta_{10}(n) + e_{10}(n)$$

where $e_{10}(n)$ is a function of the complex divisors of n, i.e. the decomposition of n in the field K(i). Similar but more complicated formulae were proved by Glaisher and Mordell for 2s = 17, 19, 16, and 18.

14. The function $\tau(n)$.

As we have shown that

(14.1) $$r_{24}(n) = \delta_{24}(n) + e_{24}(n)$$

where $\delta_{24}(n)$ depends on $\sigma_{11}(n)$ and $e_{24}(n)$ depends on $\tau(n)$, it is worth while to consider the order of magnitude of these functions. Now $\sigma_{11}(n)$ is exactly of order n^{11}, that is there exist constants A and B such that

(14.2) $$A\, n^{11} < \sigma_{11}(n) < B\, n^{11}$$

for all values of n.

By definition

(14.3) $$\sum_{n=1}^{\infty} \tau(n)\, x^n = f(x) = x\, \left[(1-x)(1-x^2)\ \ldots\ \right]^{24}$$

and using the Jacobi formula

(14.4) $$f(x) = x\left[1 - 3x + 5x^3 - 7x^6 + \ldots\right]$$

The majorant of the function in the brackets is of the form

(14.5) $$x \sum n\, x^{\alpha n^2}$$

If we put $x = e^{-y}$ this becomes

(14.6) $$\sum n\, e^{-\alpha y n^2}$$

and this is comparable with the integral

(14.7) $$\int t\, e^{-\alpha y t^2}\, dt \sim \frac{C}{y}$$

as $y \to 0$. This means that f(x) is majorised by

(14.8) $$\frac{A}{(1-x)^8}$$

as $x \to 1$. From this it follows at once that

(14.9) $$\tau(n) = O(n^8).$$

[For any function $f(x) = O\left(\frac{1}{1-|x|}\right)^8$ we calculate its coefficient a_n by Cauchy's Theorem

(14.10) $$a_n = \frac{1}{2\pi i}\int \frac{f(x)}{x^{n+1}}\, dx\ .$$

Integrating on the circle C, $|x| = 1 - \frac{1}{n}$, as $|x|^{n+1} = \left|1 - \frac{1}{n}\right|^{n+1} \sim e$, $|f(x)| = O(n)$, we obtain $a_n = O(n^8)$. But an elementary proof of Ramanujan's enables us to prove

(14.11) $$\tau(n) = O(n^7)\ .$$

He uses merely the majorant of a section of the series for f(x)

(14.12) $$\sum^{\nu} \tau(n)\, x^n \prec x\left(\sum_{\frac{1}{2}n(n+1)\le\nu} (2n+1)\, x^{\frac{n(n+1)}{2}}\right)^8$$

where the symbol $\prec$ means "is majorized by". Here in the sum on the right

$$(2n+1)^2 < 8\nu + 1$$

$$(2n+1)^2 \leqq 8\nu$$

Consequently

(14.13) $$\sum_1^{\nu} \tau(n)\, x^n \prec x\,(\sqrt{8\nu})^8\,(1 + x + x^3 + x^6 + \dots)^8$$

But here the sum raised to the eighth power enumerates the number of representations of a number as a sum of eight triangular numbers and this, like $r_8(n)$ is known to be of order n^3. Hence for $n \leqq \nu$

(14.14) $$\tau(n) < (8\nu)^4\, A\, n^3$$

whence $\tau(n) = O(n^7)$. Hardy by function theory proved

(14.15) $$\tau(n) = O(n^6)$$

(Proceedings of the Cambridge Philosophical Society, Vol. 23 (1927) pp. 675 - 684). Ramanujan conjectured that

(14.16) $$\tau(n) = O(n^{11/2+\varepsilon})$$

or more precisely that

(14.17) $$|\tau(n)| \leqq n^{11/2}\, d(n),$$

and it is known from other considerations that $d(n) = O(n^{\varepsilon})$. It may be that Hardy's result (14.15) could be sharpened by making use of Kloosterman's refinements on the number of representation of numbers by $ax^2 + by^2 + cz^2 + dt^2$ (Acta Mathematica, Vol. 49, (1926) pp. 407 - 464). The point of Hardy's method lies in studying $f(x) = x\left[(1-x)(1-x^2)(1-x^3)\dots\right]^{24}$ near the unit circle. $f(x) \to 0$ exponentially, as $x \to 1$, and also as $x \to e^{2p\pi i/q}$, but less rapidly for larger q. But on the whole $f(x)$ is large near the unit circle. If we should make a graph of $|f(x)|$ along the circle $|x| = r$ where $x = r e^{i\theta}$ for $0 \leqq \theta \leqq 2\pi$, we should find that the graph was practically level (though high) except for sharp trenches toward 0 at rational values of $\frac{\theta}{\pi}$.

Hardy shows that

(14.18) $$f(x) = O\left(\frac{1}{(1-|x|)^6}\right)$$

uniformly. Using Cauchy's Theorem (14.10), we obtain $\tau(n) = O(n^6)$.

Ramanujan conjectured, but was not able to prove that $\tau(n)$ is multiplicative, i.e.

(14.19) $$\tau(n\, n') = \tau(n)\,\tau(n') \quad \text{if } (n, n') = 1.$$

This conjecture has been proved correct by Mordell by use of the modular functions. We prove the identity

(14.20) $$\sum_1^{\infty} \tau(pn)\, x^n = \tau(p) \sum_1^{\infty} \tau(n)\, x^n - p^{11} \sum_1^{\infty} \tau(n)\, x^{pn}$$

which is interesting in itself. The function

(14.21) $$\Delta(\omega_1, \omega_2) = \left(\frac{2\pi}{\omega_2}\right)^{12} x^2 \left[(1-x^2)(1-x^4)\dots\right]^{24}$$

is invariant under the homogeneous modular group, generated by

(14.22) $$A\begin{cases}\omega_1' = \omega_1 + \omega_2 \\ \omega_2' = \omega_2\end{cases} \quad \text{and } B\begin{cases}\omega_1' = -\omega_2 \\ \omega_2' = \omega_1\end{cases}$$

corresponding to the non-homogeneous generators

(14.23) $$\tau' = \tau + 1 \quad \text{and} \quad \tau' = -\frac{1}{\tau}$$

The function

(14.24) $$\psi(\omega_1, \omega_2) = \Delta(p\omega_1, \omega_2) + \sum_{K=0}^{p-1} \Delta(\omega_1 + K\omega_2, p\omega_2)$$

is also invariant under the homogeneous modular group. $\Delta(p\omega_1, \omega_2)$ is invariant under the substitution A, while the terms of the remaining sum are merely permuted by A. If we write

(14.25) $$\psi(\omega_1, \omega_2) = \Delta(p\omega_1, \omega_2) + \Delta(\omega_1, p\omega_2) + \sum_{K=1}^{p-1} \Delta(\omega_1 + K\omega_2, p\omega_2)$$

and perform the substitution B upon ψ ,

$\Delta(pw_1, w_2) + \Delta(w_1, pw_2) \longrightarrow \Delta(-pw_2, w_1) + \Delta(-w_2, pw_1)$

$= \Delta(w_1, pw_2) + \Delta(pw_1, w_2)$. Here $\Delta(w_1 + Kw_2, pw_2)$

$\longrightarrow \Delta(-w_2 + Kw_1, pw_1) = \Delta(\Omega_1, \Omega_2)$ where $\Omega_1 = -w_2 + Kw_1$, $\Omega_2 = pw_1$. As $(p, K) = 1$, there exist solutions of $Dp + CK = 1$, and hence

(14.26)
$$\Omega_1' = p\Omega_1 - K\Omega_2$$
$$\Omega_2' = C\Omega_1 + D\Omega_2$$

is a substitution of the modular group. Performing this upon $\Delta(\Omega_1, \Omega_2)$ we obtain

(14.27)
$$\Delta(-w_2 + Kw_1, pw_1) = \Delta(-pw_2 + pKw_1 - Kpw_1, -Cw_2 + CKw_1 + Dpw_1)$$
$$= \Delta(-pw_2, w_1 - Cw_2)$$
$$= \Delta(w_1 - Cw_2, pw_2)$$

Now as $CK \equiv 1 \pmod p$ as K ranges from 1 to $p-1 \pmod p$, $-C$ must also range over the same values $(\bmod\ p)$ in some order. Hence $\sum_{K=1}^{p-1} \Delta(w_1 + Kw_2, pw_2)$ is invariant under substitution B, and consequently $\psi(w_1, w_2)$ is invariant under the homogeneous modular group.

Now from

(14.28)
$$\Delta(w_1, w_2) = \left(\frac{2\pi}{\omega_2}\right)^{12}\sum_1^\infty \tau(n)\, x^{2n}$$
$$\Delta(pw_1, w_2) = \left(\frac{2\pi}{\omega_2}\right)^{12}\sum_1^\infty \tau(n)\, x^{2pn}$$
$$\sum_{K=0}^{p-1}\Delta(w_1 + Kw_2, pw_2) = \left(\frac{2\pi}{p\,\omega_2}\right)^{12}\sum_{K=0}^{p-1}\sum_{n=1}^{\infty} \tau(n)\, x^{\frac{2n}{p}} e^{\frac{2n\pi i K}{p}}$$

it follows that $\dfrac{\Delta(w_1, w_2)}{\psi(w_1, w_2)}$ is a function of $\tau = \frac{\omega_1}{\omega_2}$ alone, as $x = e^{\pi i \tau}$, and hence must be invariant under the non-homogeneous modular group Γ. In proving this quotient bounded in D, we find the point at infinity the only debatable point. (Note that D does not reach the real axis, although D_3 does.)

Let us, in the last formula of (14.28), sum first with respect to K, and note that

(14.29)
$$\sum_{K=0}^{p-1} e^{2n\pi iK/p} = 0 \text{ if } p \nmid n$$
$$= p \text{ if } p \mid n = pm$$

Hence

(14.30)
$$\sum_{K=0}^{p-1}\Delta(w_1 + Kw_2, pw_2) = \left(\frac{2\pi}{\omega_2}\right)^{12}\frac{1}{p^{12}}\cdot p \sum_{m=1}^{\infty} \tau(pm)\, x^{2m}$$

As $\tau \to i\infty$, $x \to 0$, and in $\frac{\Delta}{\psi}$ both series begin with terms in x^2 and hence the quotient must be bounded and consequently a constant. We may now write:

(14.31)
$$C\sum_1^\infty \tau(n)\, x^{2n} = p^{-11}\sum_1^\infty \tau(pn)\, x^{2n} + \sum_1^\infty \tau(n)\, x^{2pn}$$

and comparing terms in x^2, $C = \dfrac{\tau(p)}{p^{11}}$.

We now rewrite, replacing x^2 by x.

(14.32)
$$\sum_1^\infty \tau(pn)\, x^n = \tau(p)\sum_1^\infty \tau(n)\, x^n - p^{11}\sum_1^\infty \tau(n)\, x^{pn}$$

If we now equate coefficients of $x^{sp^{\lambda-1}}$ we obtain

(14.33)
$$\tau(s\,p^{\lambda}) = \tau(p)\ \tau(s\,p^{\lambda-1}) - p^{11}\tau(s\,p^{\lambda-2}).$$

Here the term $\tau(s\,p^{\lambda-2})$ will not occur unless $s\,p^{\lambda-2}$ is integral. Let us suppose that $(s, p) = 1$ and that $\lambda = 1$. This yields

(14.34)
$$\tau(s\,p) = \tau(p)\,\tau(s)$$

For $\lambda = 2$

(14.35)
$$\tau(s\,p^2) = \tau(p)\,\tau(s\,p) - p^{11}\,\tau(s)$$

(14.35)
$$\tau(sp^2) = \tau(p)\ \tau(sp) - p^{11}\ \tau(s)$$
$$= \tau(p)^2\ \tau(s) - p^{11}\ \tau(s)$$
$$= \left\{\tau(p)^2 - p^{11}\right\}\ \tau(s)$$

But for $s = 1$ $\quad \tau(p^2) = \tau(p)^2 - p^{11}$

and hence

(14.36)
$$\tau(sp^2) = \tau(p^2)\ \tau(s)$$

Similarly, by induction

(14.37)
$$\tau(sp^\lambda) = \tau(p^\lambda)\ \tau(s) \qquad \text{if } (s, p) = 1$$

and hence generally

(14.38)
$$\tau(mn) = \tau(m)\ \tau(n) \qquad \text{if } (m, n) = 1$$

proving the multiplicative property of $\tau(n)$.

From the multiplicative property of $\tau(n)$ it follows that the Dirichlet series

(14.39)
$$f(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$$

may be factored into

(14.40)
$$f(s) = \prod_p \left(1 + \frac{\tau(p)}{p^s} + \dots\right)$$

Using (14.33) as a scale of relation

(14.41)
$$1 + \frac{\tau(p)}{p^s} + \frac{\tau(p^2)}{(p^s)^2} + \frac{\tau(p^3)}{(p^s)^3} + \dots$$
$$= \frac{1}{1 - \tau(p)\ p^{-s} + p^{11-2s}}$$

and hence

(14.42)
$$f(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \tau(p)\ p^{-s} + p^{11-2s}} .$$

If we write

(14.43)
$$\cos \theta_p = \tfrac{1}{2} p^{-11/2}\ \tau(p)$$

then

(14.44)
$$\tau(p^\lambda) = p^{11/2} \frac{\sin(\lambda+1)\theta_p}{\sin \theta_p}$$

if we appeal to the well known identity

(14.45)
$$\frac{2\ r \sin\theta}{1 - 2\ r \cos\theta + r^2} = 2\ r \sin\theta + 2\ r^2 \sin 2\theta + \dots$$

and hence if $n = \prod p^\lambda$

(14.46)
$$\tau(n) = n^{11/2} \prod_{p/n} \frac{\sin(\lambda+1)\theta_p}{\sin \theta_p}$$

It is extremely probable that θ_p is always real, i.e.

(14.47)
$$|\tau(p)| < 2\ p^{11/2}$$

but this has never been proved. The truth of (14.47) implies the truth of (14.17), of which it is the special case when n is a prime. If θ_p is real, and $n = p^\lambda$, $\tau(n) = n^{11/2} \dfrac{\sin(\lambda+1)\theta_p}{\sin \theta}$. We can choose λ so that $\lambda\theta_p$ is as near as we please to a multiple of π, and then $|\tau(n)| >$ const $n^{11/2}$. On the other hand if θ_p is complex, then $|\tau(p)| = |p^{11/2}\ 2 \cos\theta_p| > p^{11/2}$. Hence in any case $|\tau(n)| >$ const. $n^{11/2}$ for infinitely many n, and $11/2$ is a 'best possible' index.

Hardy, using function theoretical methods, has proved a number of properties of $\tau(n)$ and $f(s)$. There exist absolute constants, A and B such that

(14.48)
$$A\ n^{12} < \left[(\tau(1))^2 + (\tau(2))^2 + \dots + (\tau(n))^2\right] < B\ n^{12}$$

whence the average order of $\tau(n)$ is $n^{11/2}$. This also shows that (14.16), if true, is a best possible result.

The function $f(s)$ has many properties similar to those of $\zeta(s)$, and almost every problem on $\zeta(s)$ has an analogue for $f(s)$. The series (14.39) is absolutely convergent for $\sigma > 13/2$ and defines, by analytic continuation an entire function. There is a functional equation satisfied by $f(s)$ even simpler than that for $\zeta(s)$. It is

(14.49) $$(2\pi)^{-s}\,\Gamma(s)\,f(s) = (2\pi)^{s-12}\,\Gamma(12-s)\,f(12-s)$$

From (14.39) and (14.49) we may determine $f(s)$ for $\sigma < 11/2$ or for $\sigma > 13/2$, but as in the case of $\zeta(s)$ there is a critical strip in which its behaviour is mysterious. To prove (14.49) we observe that

(14.50) $$\Gamma(s)\,f(s) = \int_0^\infty y^{s-1}\,F(e^{-y})\,dy$$

where

(14.51) $$F(e^{-y}) = e^{-y}\,(1-e^{-y})^{24}\,(1-e^{-2y})^{24}\ \ldots$$

Now (14.50) is valid for large s, hence for all s, and so $f(s)$ is an entire function. Moreover

(14.52) $$F(e^{-y}) = \left(\frac{2\pi}{y}\right)^{12} F(e^{-4\pi^2/y})$$

and this we substitute into (14.50), replace $4\pi^2/y$ by z giving

(14.53) $$\Gamma(s)\,f(s) = (2\pi)^{12}\,(2\pi)^{2s-24}\int_0^\infty z^{11-s}\,F(e^{-z})\,dz$$

and this with (14.50) leads immediately to the functional equation (14.49).

There is a Riemann-Hypothesis for $f(s)$, namely that all its zeros are on $\sigma = 6$. For $f(s)$, J. R. Wilson has proved the theorem which Hardy proved for $\zeta(s)$, that there are infinitely many zeros on the line where we hope they all are. The analogue of the prime number theorem for $f(s)$ is

(14.54) $$\sum_{p\leq n}\tau(p)\log p = o(n^{13/2})$$

and this is probably equivalent to $f(13/2+ti) \neq 0$.