CAAP'86
LNCS, Vol. 214, 1986, pp 6-16

COLOURINGS OF PLANAR MAPS

AND THE EQUALITY OF TWO LANGUAGES

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The investigation of the four coulour problem inspired the developement of a lot of important combinatorial theories ; in the paper of T. Saaty [6] thirteen statements equivalent to the existence of a four colouring for planar maps are quoted. These statements are not only concerned with graph theory and some of thems seem far from graphs like those looking as number theory problems. In what follows, we add a statement in the field of formal language theory. We show that the 4-colouring of planar maps is equivalent to the equality of two subsets in the cartesian product of two free monoïds. Unfortunately this doesn't give immediately new results in the field of map colouring as the question of the equality of two subsets in the cartesian product of two free monoïds is undecidable.

Also, the fact that any planar map has a five coulouring, which is not so difficult to prove, has a formal language theory version which seems untractable.

In order to transform four colourings of maps into words of a free monoïd we use the intermediate notion of tree. The first construction we give was proposed by F. Jaeger, it allows the decomposition of a cubic planar map into two trees, that we discribe in part I. The next step consists in coding a tree by a word of the Lukasiewicz language giving the way to transform problems on trees into problems on words.

This communication is concerned with maps, trees and words ; it is difficult to give all the definitions concerning these notions, we will only restate a few of thems. The books of O. Ore [5], D.Knuth [2], M. Lothaire [3] and J. Berstel [1] contain more detailed presentation of maps, trees, words and formal languages respectively.

I - Planar maps and flows

Let us state a few definitions concerning maps. A planar map determines a partition of the plane in a set S of vertices, a set A of edges and a set F of faces . Each vertex is a point of the plane, each edge is an open curve having two vertices as end points and each face is an open simply connected domain bounded by edges and vertices. Each edge is in the boundary of two faces ; in the sequel we suppose that these faces are distinct thus the map has no isthmus.

A <u>colouring</u> of the faces of a map consists in a mapping of F in a finite set of colours in such a way that any two faces having a common edge in their bourdaries are mapped into different colours.

The <u>degree</u> of a vertex of a map is equal to the number of edges having it as end point. A map is said <u>cubic</u> if all its vertices have degree three.

An <u>orientation</u> of a map is determined by choosing for each edge a an <u>initial end point</u> i(a) and a <u>terminal end point</u> t(a).

A <u>k-flow</u> on an oriented map is a mapping φ of the set of edges into the ring $\mathbb{Z}/k\mathbb{Z}$ of integers modulo k such that, for any vertex s, we have :

$$\sum_{i(a)=s} \varphi(a) \equiv \sum_{t(a)=s} \varphi(a) \qquad (\mathrm{mod}\ k)$$

A <u>non-zero</u> k-flow is a flow φ for which $\varphi(a)$ is different from 0 for any edge a. In fact, it is better to say a nowhere zero k-flow, but non zero is shorter.

Remark that the existence of a non-zero k-flow for an oriented map depends only on the map itself and not on the orientation choosen. As if M' is obtained from M by changing the orientation of the edge a_0 then, if φ is a non-zero k-flow on M, the mapping φ' defined below is a non-zero k-flow on M' :

$$\varphi'(a) = \varphi(a) \quad \text{for each} \quad a \neq a_0 \quad \text{and} \quad \varphi'(a_0) = -\varphi(a_0) .$$

If an oriented map has a colouring consisting in a mapping of the set of faces in {1,2,...,k} then one can construct a non-zero k-flow by defining φ(<u>a</u>) as the difference (molulo k) between the coulours of the faces containing <u>a</u> in their boundary (the orientation of <u>a</u> allows to distinguish between the face at the left of <u>a</u> and that at the right of <u>a</u>). In fact, this construction can be reversed giving the following theorem which seems to be due to W.T. Tutte [7].

Theorem 1 : The following statements are equivalent :

(1) Every planar map with no isthmus is face colourable in four colours.

(2) Every planar map with no isthmus has a non-zero 4-flow.

(3) Every cubic planar map with no isthmus has a non-zero 4-flow.

II- Flows in binary trees :

In what follows we consider that a binary tree is determined by a set of nodes, one of them being the root r, such that each node, except the root, has two sons or no son at all ; between the two sons of a node one is distinguished as the left son, the other being the right one. A node which has no son is called a leaf. Generally, the root of a binary tree has two sons, in order to simplify further developments we suppose that it has only one son (or no son if the tree reduces to r). The size of a binary tree is equal to the total number of nodes ; with our conventions concerning the root this size equals twice the number of leaves of the tree.

Figure 1 gives an example of a binary tree of size 12.

The left (resp. right) subtree below node n is recursively defined in the following way :

If n is a leaf then this left (resp. right) subtree is empty, if it is not a leaf then it consists of the left son of n (resp. the right son of n) and of the left and right subtrees below this left (resp. right) son.

Below the root there is a subtree consisting of all the other nodes.

The preorder sequence of the vertices of a tree is obtained by concatenating :

the root r, its son s, the preorder sequence of the left subtree below s, the preorder sequence of the right subtree below s. This sequence determines a total order on the set of nodes, in the sequel when we will speak of i^{th} leaf this will mean the i^{th} leaf in the preorder sequence. In figure 1 the nodes are numbered as they appear in the preorder sequence : the first leaf is numbered 3, the second one is numbered 6 and the 6^{th} one 12.

A k-flow in a binary tree is given by a mapping Ψ of the set of nodes in the set Z/kZ of integers moludo k, such that the flow in a node which is not a leaf is equal to the sum (modulo k) of the flows of its sons. Note that a flow is completely determined by its value on the set of leaves . A flow Ψ is non zero if $\Psi(n)$ is different from zero for any node n. Figure 2 gives a non zero 4-flow for the tree in figure 1.

Given two binary trees $\mathscr{A}_1$ and $\mathscr{A}_2$ of the same size, two flows Ψ_1 on $\mathscr{A}_1$ and Ψ_2 on $\mathscr{A}_2$ are said to be compatible if the flows on the roots are the same $(\Psi(r_1) = \Psi(r_2))$ and if the flow on the i^{th} leaf of $\mathscr{A}_1$ is equal to the flow on the i^{th} leaf of $\mathscr{A}_2$ for all leaves. Figure 3 gives a binary tree of same size than that of figure 2 and a flow compatible with that previously given.

Theorem 2 ([4], see also [6]) - The two following statements are equivalent :

(1) Every planar map with no isthmus is face colourable in 4 colours.

(4) Two binary trees of the same size have two non-zero 4-flows which are compatible.

Proof. We use the condition (3) of theorem 1. We consider here planar cubic maps with no isthmus such that any union of two or three faces (with their boundaries) which are two by two adjacent (i.e. with a common boundary edge) is a simply connected domain. Then, it' s easy to prove that if any planar cubic map with no isthmus verifying this condition is face colourable in four colours (or admits a non-zero 4-flow), it's the same for any planar cubic map with no isthmus. Let C be a planar cubic map with no ithmus verifying the previous condition and let C^* be the dual of C (its vertices correspond to the faces of C and two vertices are joigned by an edge if the corresponding faces have a boundary edge in common). In C^* all the faces are bounded by three edges as C was cubic and there is no cycle which contains one, two or three edges excepted the cycles composed by the boundaries of the faces which are triangles. Thus by a theorem of Whitney [8], C^* has a hamiltonian cycle.

This hamiltonian cycle of C^* determines in C a closed curve Γ which go through any face cutting some edges : thus C is decomposed in a binary tree in the interior of Γ and a binary tree in its exterior. The leaves of these binary trees are the points in which Γ intersects the edges of C, and a root is choosen among them.

Then, it is clear that the existence of a non-zero 4-flow for C is equivalent to that of two compatible non zero 4-flows for the trees obtained in the decomposition :

The flow of a leaf of the trees is equal to the flow in the corresponding edge intersected by Γ, and the flow (in the tree) of the other nodes is determined by the flow (in the map) in the edge joining it to its "father". This construction can clearly be reversed. □

III - Coding binary trees

Let X and Y be the following alphabets :

$$X = \{x, \bar{x}\} \quad Y = \{x, x_1, x_2, x_3\}$$

To any binary tree is associated a word of X^* considering its preorder sequence of nodes and writing an $\bar{x}$ for any leaf and an x for a node which is not a leaf, nothing being written for the root. Clearly the word corresponding to a tree of size $2k$ has length $2k-1$, k occurrences of $\bar{x}$ and $k-1$ of x. Moreover we have :

Proposition 1 - The set of words associated with binary trees is a context free language L generated by the grammar :

$$\xi \to x\xi\xi \qquad \xi \to \bar{x} .$$

Proof. This proposition is deduced from the fact that the coding process considers first the son s of the root;then,if s is not a leaf, the left subtree below this son, then the right subtree below it (explaining the first rule) ; when s is a leaf then the code is $\bar{x}$ (explaining the second rule).

In order to code a binary tree with a flow it is only necessary to remind the value of this flow on the leaves, as remarked in paragraph II. Then we use the following process : from the preorder sequence of the vertices of the tree, write x for any node which is not the root and x_i for a leaf having a flow equal to i.

This gives :

Proposition 2 - the set of words coding binary trees having a non zero 4-flow with value equal to 1 on the root is a context free language L_1 given by the grammar (ξ_1 is the axiom) :

$$\xi_1 \to x\ \xi_2\ \xi_3 \quad \xi_1 \to x\ \xi_3\ \xi_2 \quad \xi_1 \to x_1$$

$$\xi_2 \to x\ \xi_1\ \xi_1 \quad \xi_2 \to x\ \xi_3\ \xi_3 \quad \xi_2 \to x_2$$

$$\xi_3 \to x\ \xi_1\ \xi_2 \quad \xi_3 \to x\ \xi_2\ \xi_1 \quad \xi_3 \to x_3 .$$

Proof - Similar considerations to that given for the proof of proposition 1 give the result. It suffices to note that the words generated using axiom ξ_i code the binary trees having a flow with value i on the root ; and that $\xi_i \to x\ \xi_j\ \xi_k$ is a rule then $j+k \equiv i$ (modulo 4) as the flow in a node is equal to the sum of the flows of its sons.

IV – A language in $X^* \times X^*$

In this part we use the definitions and notations of [1].
Let φ be the morphism of Y^* in itself defined by $\varphi(x) = 1$, $\varphi(x_i) = x_i$: φ forgets the x's and conserves the x_i. This morphism can be extended in a morphism of $Y^* \times Y^*$ in itself by :

$$\varphi((f,g)) = (\varphi(f), \varphi(g)).$$

Let Ψ be the morphism of Y^* in X^* defined by $\Psi(x) = x$ and $\Psi(x_i) = \bar{x}$ for $i=1,2,3$: Ψ replaces all the letters x_i by $\bar{x}$, thus if f is an element of L coding a binary tree with a 4-flow then $\Psi(f)$ code the same binary tree forgetting the flow.

Let Δ be the subset of $Y^* \times Y^*$ consisting of all the pairs (u,u) ; clearly Δ is a rational subset of $Y^* \times Y^*$ as :

$$\Delta = \{(x_i, x_i) \mid i=1,2,3\}^* .$$

Let D be the subset of $X^* \times X^*$ consisting of all the pairs (f,g) such that $|f| = |g|$, D is also a rational subset of $X^* \times X^*$ as

$$D = \{(x,\bar{x}) , (\bar{x},x), (x,x), (\bar{x},\bar{x})\}^*$$

<u>Theorem 3</u> – The two following statements are equivalent :

(1) Every planar map with no isthmus has a face colouring in four colours.

(5) The two following subsets of $X^* \times X^*$ are equal :

$$P = \Psi\ (\varphi^{-1}(\Delta) \cap (L_1 \times L_1))$$
$$Q = (L \times L) \cap D.$$

<u>Proof</u>. Of course we will show that (5) is equivalent to the statement (4) of theorem 2.

Two binary trees $\mathcal{A}_1$ and $\mathcal{A}_2$ of the same size are coded by two words f and g such that $|f| = |g|$ thus (f,g) is an element of Q.

A non zero 4-flow on a binary tree coded by f gives a code u of Y^* such that $\Psi(u) = f$; the same is true for a binary tree coded by g giving v such that $\Psi(v) = g$. The order in which are met the x_i in u and in v is the same as that of the leaves in the preorder sequences of the vertices of the two trees ; the two 4-flows are thus compatible if $\varphi(u) = \varphi(v)$ or equivalentely if $\varphi((u,v)) \in \Delta$.

Thus two binary trees coded by f and g admit compatible non-zero 4-flows if and only if it exists two words u and v of L such that

$$\Psi(u,v) = (f,g) \quad ; \quad \varphi(u,v) \in \Delta$$

which ends the proof.

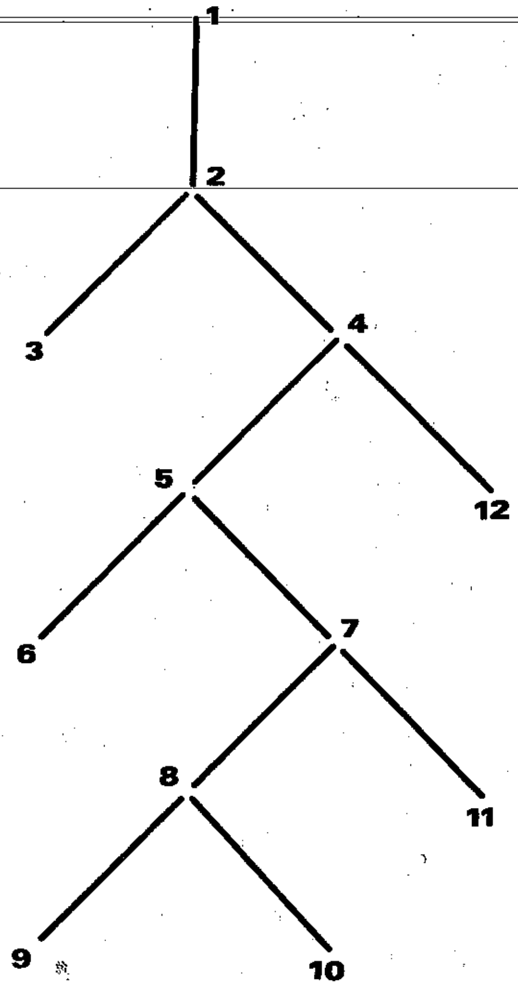

Figure 1 : A binary tree of size 12 in which the vertices are numbered as they appear in the preorder sequence.

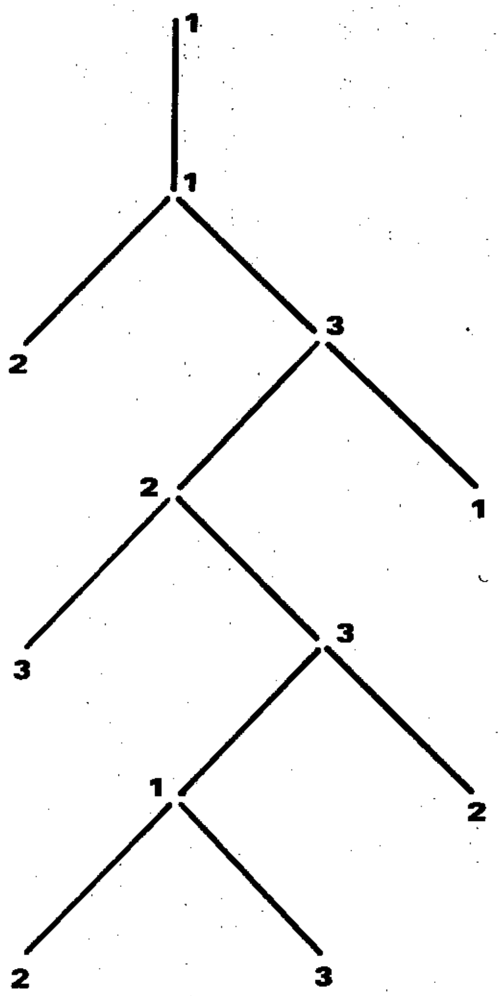

Figure 2 : A non-zero 4-flow for the tree given in figure 1.

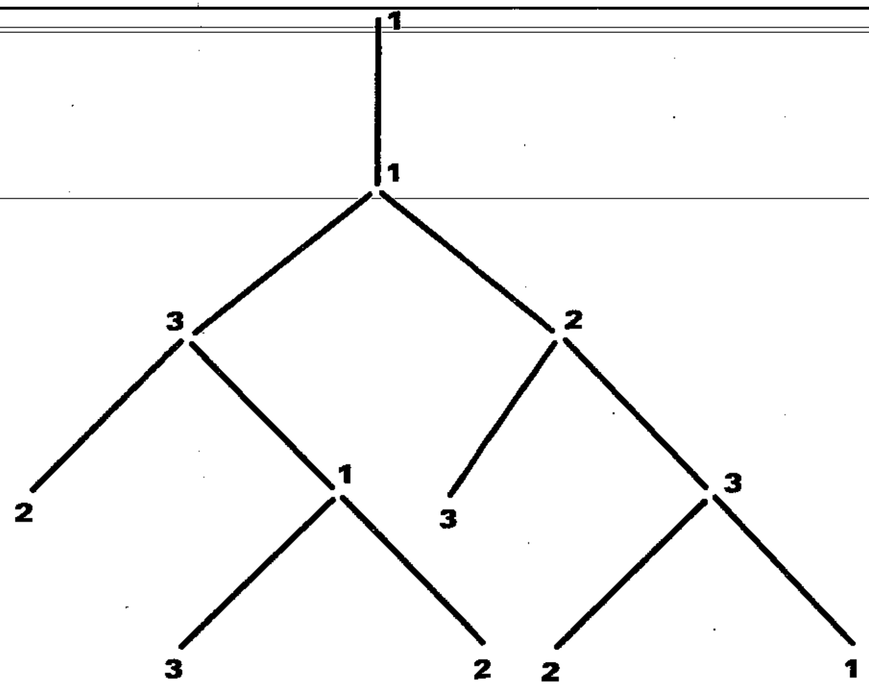

Figure 3 : A 4-flow compatible with that given in figure 2 for a tree of the same size.

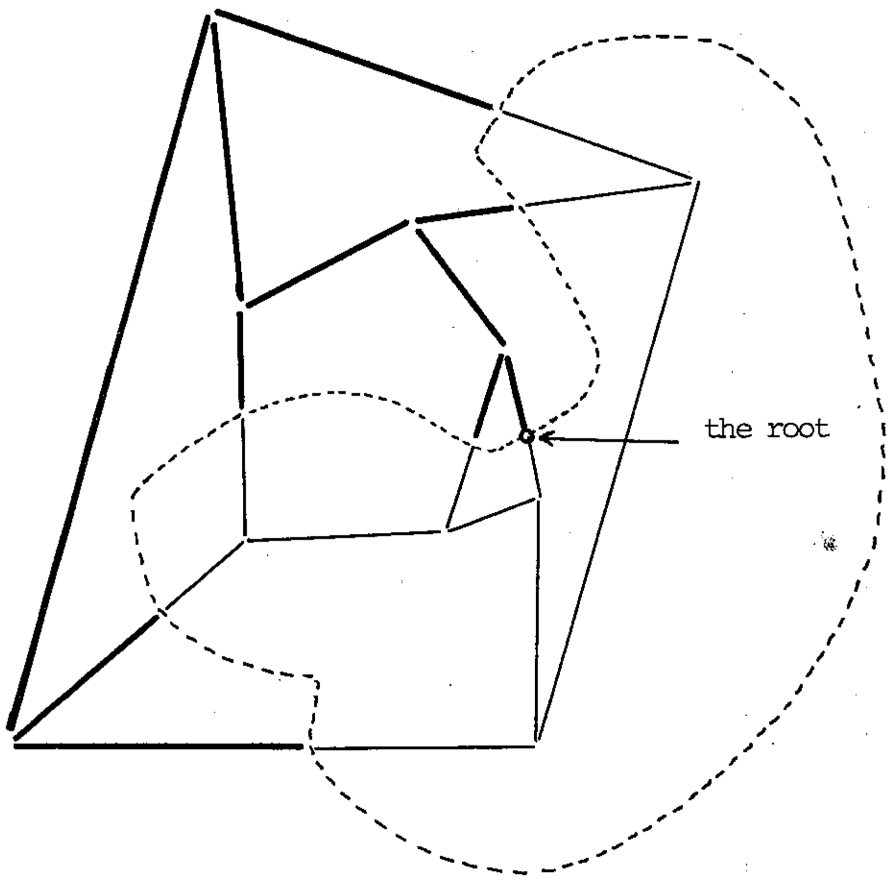

Figure 4 : A cubic map C and the trees obtained by its decomposition using a hamiltonian cycle of C^*. Note that the two trees obtained are those of figures 2 and 3.

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