USING EXPERIMENTAL MATHEMATICS TO CONJECTURE AND PROVE THEOREMS IN THE THEORY OF PARTITIONS AND COMMUTATIVE AND NON-COMMUTATIVE RECURRENCES

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A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

The principal advisor's name

and approved by

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New Brunswick, New Jersey

May, 2016

ABSTRACT OF THE DISSERTATION

Using experimental mathematics to conjecture and prove theorems in the theory of partitions and commutative and non-commutative recurrences

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Dissertation Director: The principal advisor's name

This is the abstract

Acknowledgements

The body of the acknowledgements

Dedication

This is the dedication

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Chapter 1

Introduction

The underlying theme for this thesis is "experimental mathematics". We use this term to denote the use of computers in mathematical research. Over the past several decades, experimental mathematics has been a burgeoning field. Computers were the key to Appel and Haken's proof of the four-color theorem, along with Thomas Hales's solution of the Kepler conjecture. In both works, the problem was reduced to checking a very large number of cases.

Mathematics is one of the most interesting academic disciplines, as it lies at the intersection of art and science. Most sciences have some experimental aspect to them, and it is interesting to see in what ways this facet can be seen in mathematics. Much of my work could be called carrying out "mathematical experiments" — trying a large number of possibilities and examining the results. Other features of experimentation sometimes arise in mathematics. For example, the integrality of the Dana Scott sequence was discovered "by accident" — Scott was attempting to program the Somos-4 sequence, a recursively defined sequence that surprisingly only contains integers, into a computer to study it. However, he made a typing error while inputting the recurrence. As a result, Scott serendipitiously made the discovery that the same phenomenon that holds true for the Somos-4 sequence also occurs in this new sequence with the typo.

The practice of experimental mathematics can take many different forms. We now highlight a few selected avenues, along with corresponding examples from this thesis.

In Chapters — and —, I use Maple code to automatically search for and discover new (conjectural) partition identities of Rogers-Ramanujan type.The main point of involving the computer in this process is to speed up calculations that could otherwise be done by hand. For example, when I use computers to search for new partition

identities, I am just determining the first 30 or so terms of the formal power series for a given "sum side", and then seeing if it can be factored as a "nice" infinite product. Now, I could probably do the calculations for any given sum side using pen and paper, and sometimes, it is important to do some work myself to better be able to spot patterns. However, this undertaking would probably take me an entire afternoon. If I were to use computers instead, I can do the calculations for, say, a million sum sides in the same amount of time. One can only imagine what Ramanujan would have accomplished had he had access to modern computers. Using computers also has the advantage of being more accurate. Because I am doing exact calculations, miscounting by a single partition would throw off my entire calculations. Others have used computers to search for partition identities in the past, but our "sum-side" conditions are more apparently more general than what have been used in the past.

We can do more than simply use computers to conjecture results and involve them in the proof process as well. Examples of this fill the next several chapters.

In the late 1980s, Andrews and Baxter published what they called a "motivated proof" of the Rogers-Ramanujan identities. The key observation is that a sequence of power series is constructed, beginning with the two Rogers-Ramanujan product sides, and each term in the sequence is noted to be of the form $1 + q^j + \cdots$ for ever-increasing values of j. They called this observation their "Empirical Hypothesis".

Since this work, a program has been developed to apply this "motivated proof" paradigm to other partition identities, including a project I worked on with Kanade, Lepowsky, and Sills to provide a similar proof of the Andrews-Bressoud identities. Once again, an "empirical hypothesis" was deduced. In our case, it truly was empirical — Maple was used to calculate the first several dozen terms of the initial power series on what we call the "zeroth shelf", and then experimentation was used to find appropriate linear combinations of these power series on the "zeroth shelf" to make up a "first shelf", and then a "second shelf", and so on, always verifying that the elements of each "shelf" are always power series of the form $1 + q^j + \cdots$. Once the correct linear combinations are determined, it is then relatively straightforward to uncover the underlying interpretation through partitions.

Even if we are unable to use computers to provide complete proofs, we can use them to guide our efforts to provide "human" proofs. These efforts are evident in the chapters of my work on demonstrating that certain noncommutative recursions possess the Laurent phenomenon — that every term of the recursively-defined sequence is a Laurent polynomial in the initial variables. The main theorem is shown to hold for any noncommutative recursion of a certain form using monic palindromic polynomials in $\mathbb{Q}[x]$, generalizing a conjecture by Kontsevich. One way to prove the theorem, in the spirit of Berenstein and Retakh, is by explicitly calculating the first term in the recursion that could possibly not be a Laurent polynomial, and by demonstrating, through extensive algebraic manipulations, that it is, in fact, a Laurent polynomial. The form of these long calculations, where it is easy for a human to err, make it natural to turn to a computer for assistance. Accordingly, I wrote a Maple package that allowed me to prove the theorem for any fixed set of monic palindromic polynomials. The computer is taught a list of "simplifying" rules, which gradually transform the expression in question into a particular form. At the end of all of the calculations, the expression can be observed to be a Laurent polynomial, which verifies the theorem in that case.

Of course, writing this computer code is not enough to prove the theorem. I then used the output of the program for certain cases of the theorem to guide my proof, which is contained in chapter —. The proof is rather technical, consisting of a long series of calculations, in the style of the computer. Watching the steps that my procedures take to verify specific instances of the theorem allowed me to decide what intermediate steps needed to be done.

In the rest of the chapter, I use what I have learned to extend this to a family of two-dimensional recurrences. Again, having "practiced" by seeing the output of the Maple procedures greatly facilitates the work by hand.

Perhaps the purest form of experimental mathematics is having the computer discover and then automatically prove new theorems. This paradigm is explored in Chapter —. In this chapter, Somos-like sequences are discussed. These are the commutative versions of the recursions mentioned in the previous paragraph. Again, the goal is to

establish the Laurent phenomenon for specific sequences. If the initial variables of the sequences are all specialized to 1, then the Laurent phenomenon becomes the statement that every term in the sequences is an integer.

For a large number of recursions, chosen in a way that made them likely candidates, I used Maple to first evaluate if the underlying sequence with the all-ones initial conditions only contains integers for the first 15 terms or so. (Of course, we could do this without specializing the variables to all be equal to 1, but using integers instead of formal variables greatly speeds up the running time.)

This process indentified likely candidates for sequences with the Laurent phenomenon. However, we can do even more. Fomin and Zelevinsky provided machinery to prove the Laurent phenomenon for specific sequences that is very amenable to automation. I implemented this machinery in Maple in such a way that the computer is able to nearly simultaneously conjecture and prove new results.

Although the most important part of the work is, again, the computer code used to discover the results, I spend part of the chapter proving the Laurent phenomenon for certain infinite families of recursions, guided by the experimental data.

Although the mathematical fields I investigated in this thesis may be quite diverse, the thread that ties this work together is more about the underlying methods used in experimental mathematics. Note how in both some of my explorations in partition theory and also in these Somos-like sequences, the approach is to calculate some number of initial terms of a given sequence (in the case of partition identities, the sequence at hand is the sequence of coefficient of the formal power series on the sum side), and then check that all of the terms satisfy some condition.

Often, when people think of using computers in mathematics, "number-crunching" is what comes to mind - long calculations in arithmetic designed to obtain approximate answers. Instead, my work tends to be in areas where "symbol-crunching" is more useful, involving long algebraic manipulations. Furthermore, I need exact answers in my work — as noted earlier, being off by a single partition would throw off my entire calculation.

It would be wonderful if more mathematics could be done by way of what Zeilberger

calls the N_0 principle. This framework of proof is when one checks that a theorem holds in a certain number of cases, and then argues that the truth of the theorem for these cases suffices to prove that the truth of the general theorem. Going back to the partition identity conjectures, it certainly seems that we have enough cases to believe in their truth. Again, using Maple (along with a bit of human ingenuity), we were able to verify our conjectures to hold for all $n \leq 1500$ (at the minimum). MacMahon famously was convinced of the truth of the Rogers-Ramanujan identities after observing that they held for about 90 terms. Unfortunately, right now, no such "meta-theorem" exists. However, our new partition conjectures are leading to new research as others attempt to understand them more deeply and prove them.

Chapter — is joint work with Shashank Kanade. Chapter — is joint work with Shashank Kanade, James Lepowsky, and Andrew V. Sills.

Chapter 2

Experimental mathematics and partition identities

2.1 Introduction

The work of this chapter is an attempt to construct a mechanism using computer algebra systems to discover and conjecture new partition identities. Six new conjectured partition identities are the major result of this chapter. The Maple code for the work in this chapter and the following can be found in the package `IdentityFinder`, freely available at `http://math.rutgers.edu/~russell2/papers/partitions14.html`.

These identities are in the spirit of many other well-known identities, such as Euler's identity, the Rogers-Ramanujan identities, Schur's identity, Gordon's identities, the Andrews-Bressoud identities, Capparelli's identities, and the little Göllnitz identities. Using a systematic mechanism based on symbolic computation, we present six new conjectured identities using `IdentityFinder`.

The main idea behind `IdentityFinder` is to calculate out a very large number of sum side generating functions, followed by the use of Euler's algorithm to turn the generating functions into products. At this point, we examine the products to see which ones provide "periodic" factorizations. If this happens (with a small enough period), we identify the pair of the sum side and product side as a potential new partition identity.

Each sum side incorporates various restrictions on partitions, such as difference-at-a-distance conditions, congruence-at-a-distance conditions, initial conditions, etc., each of which conditions are coded as separate procedures. We hope that the "modular" nature of our work will allow us to extend our work to incorporate other possible sum side structures, allowing for new advances. A few selected avenues of research are highlighted at the end of this chapter.

Using computer explorations to discover and prove partition identities is not new. Similar methods to ours have been previously used effectively by Andrews (see [7] and Chapter 10 of [6]). Computer searches have been used extensively in automatically discovering finite Rogers-Ramanujan type identities (see [56] and [47]) and analytic identities (see [47]). See the introduction of [47] for other references for computer methods in partition identities.

Apart from their intrinsic interest, partition identities are fundamentally related in deep ways to have connections to divers fields of mathematics. For example, Baxter [13] demonstrated how the Rogers-Ramanujan identities are useful in statistical mechanics in his solution of the hard-hexagon model. Also, Lepowsky and Wilson [38]–[41] found connections between many partition identities (including the Rogers-Ramanujan identities) and affine Lie algebras in the realm of vertex operator algebras. Since then, some new partition identities have been found through Lie-theoretic approaches, such as Capparelli's identities [16]. We believe that some of our conjectures $(I_1 - I_3)$ are connected with the level 3 standard modules of the affine Lie algebra $D_4^{(3)}$. (These feature symmetric congruence classes for the product sides; we should not expect Lie-theoretic connections when the congruence classes are asymmetric.)

While we do not have formal proofs of these conjectures, we strongly believe in their truth. All six conjectures in this section have been verified up to at least partitions of $n = 1500$ (through the polynomial recursions presented in section ——).

2.2 Preliminaries

Throughout this work, we will treat q as a formal variable. Let n be a non-negative integer. A partition of n is a list of integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ (which we will often write as $\lambda_1 + \lambda_2 + \cdots + \lambda_m$) such that $\lambda_1 + \cdots + \lambda_m = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1$. We refer to the individual λ_is as "parts". We assume that $n = 0$ has exactly one partition, which is the null partition.

One obvious question related to partitions is to count the number of partitions of a nonnegative integer n. We can also place restrictions on the partitions that we are

considering. For example, we may desire to only count partitions where the parts are in some residue class modulo some N — for example, partitions into odd parts.

A partition λ is said to contain a second partition μ as a subpartition if $(\mu_1, \mu_2, \ldots, \mu_\ell)$ is a subsequence of $(\lambda_1, \lambda_2, \ldots, \lambda_m)$. In our work below, our initial conditions will typically be of the form of a small set of forbidden subpartitions.

To us, a "product side" is a generating function of the form $\prod_{j \geq 1} (1 - q^j)^{p_j}$. Usually, each p_j will be in $\{0, 1\}$, which means that this expression will be the generating function for partitions, where the allowable parts are those j such that $p_j = -1$. Furthermore, if $\{p_j\}$ is a periodic sequence, then the product can be interpreted as the generating function for partitions whose parts satisfy certain congruence conditions modulo the period.

However, the "sum side" involves difference conditions between parts — how the parts interact with each other — along with initial conditions .(forbidding a small number of subpartitions). As an example, the sum side of Euler's identity is that all parts must be distinct: $\lambda_i > \lambda_{i+1}$ for all i.

A "partition identity" is a statement that, for all nonnegative integers n, the number of partitions of n that satisfy the product side conditions equals the number of partitions of n that satisfy the sum side conditions.

Euler's algorithm allows us a way to obtain a product side from a sum side by "factoring":

Proposition 2.2.1. Let $f(q)$ be a formal power series such that

$$f(q) = 1 + \sum_{n \geq 1} b_n q^n. \tag{2.1}$$

Then

$$f(q) = \prod_{m \geq 1} (1 - q^m)^{-a_m}, \tag{2.2}$$

where the a_ms are defined recursively by:

$$n b_n = n a_n + \sum_{d \mid n,\, d < n} d a_d + \sum_{j=1}^{n-1} \left(\sum_{d \mid j} d a_d \right) b_{n-j}. \tag{2.3}$$

For a proof using logarithmic differentiation, see Theorem 10.3 of [6].

If we have a generating function $f(q)$ having constant term 1, we refer to the form (2.1) as the "sum side" and to (2.2) as the "product side."

Of course, our methods cannot actually produce genuinely infinite formal power series. Rather, we end up with polynomials that approximate these infinite formal power series. The following corollary implies that if we provide additional terms of our generating function, obtaining a better approximation, the original factors in the original product are unchanged. This will be used in our verification section.

Corollary 2.2.2. Let $f(q)$ and $g(q)$ be formal power series with constant term 1 such that

$$f(q) - g(q) \in q^{k+1}\mathbb{C}[[q]]$$

for some $k \geq 1$. If

$$f(q) = \prod_{m \geq 1} (1 - q^m)^{-a_m^{(f)}} \tag{2.4}$$

and

$$g(q) = \prod_{m \geq 1} (1 - q^m)^{-a_m^{(g)}} \tag{2.5}$$

then $a_m^{(f)} = a_m^{(g)}$ for all $m = 1, \ldots, k$.

Proof. Equation (2.3) implies that a_n is dependent only on $b_1, \ldots, b_n$. $\qquad\square$

2.3   History

The history of partition identities is a particularly fascinating one. We will give a brief overview; for more information, see [6].

The oldest partition identity is due to Euler. It states that for any non-negative integer n, the number of partitions of n only using odd parts is the same as the number of partitions of n in which all parts are distinct. This identity was originally stated and proved in its generating function form:

$$\prod_{j \geq 1} \frac{1}{(1 - q^{2j-1})} = \prod_{j \geq 1} \left(1 + q^j\right), \tag{2.6}$$

This is the archetypal example of a partition identity. One of the restricted types is into parts that fall into certain congruence classes (typically referred to as the "product side", as the generating function is most easily expressed as an infinite product), while the other deals with how parts interact with each other (which we will call the "sum side", because its generating function is often written as an infinite sum — although not in this case). Loosely speaking, a partition identity is a statement that, for all non-negative integers n, the number of partitions of n into partitions of one restricted type is the same as the number of partitions of n into partitions of a different restricted type.

The proof of Euler's identity is rather straightforward. One simply manipulates the generating functions in the following way:

$$\prod_{j \geq 1} \frac{1}{\left(1 - q^{2j-1}\right)} = \prod_{j \geq 1} \frac{\left(1 - q^{2j}\right)}{\left(1 - q^{2j-1}\right)\left(1 - q^{2j}\right)} \tag{2.7}$$

$$= \prod_{j \geq 1} \frac{\left(1 - q^{j}\right)\left(1 + q^{j}\right)}{\left(1 - q^{j}\right)} \tag{2.8}$$

$$= \prod_{j \geq 1} \left(1 + q^{j}\right), \tag{2.9}$$

where we make use of the identity $\left(1 - x^2\right) = \left(1 - x\right)\left(1 + x\right)$.

Euler's identity is the only one on this list that can be proven by using "manipulatorics" — all of the other identities presented in this chapter have very deep proofs (and so we should expect any proof of the conjectures to be appropriately complicated).

The next most complicated identities are the Rogers-Ramanujan identities, originally discovered by L. J. Rogers in 1894, and rediscovered by Ramanujan in the 1910s:

- The number of partitions of a non-negative integer into parts congruent to ± 1 (mod 5) is the same as the number of partitions with difference at least 2.

- The number of partitions of a non-negative integer into parts congruent to ± 2 (mod 5) is the same as the number of partitions with difference at least 2, and 1 is not allowed as a part.

In generating function form, these are:

$$\prod_{j\geq 0} \frac{1}{\left(1 - q^{5j+1}\right)\left(1 - q^{5j+4}\right)} = \sum_{n\geq 0} d_1(n)q^n, \tag{2.10}$$

$$\prod_{j\geq 0} \frac{1}{\left(1 - q^{5j+2}\right)\left(1 - q^{5j+3}\right)} = \sum_{n\geq 0} d_2(n)q^n, \tag{2.11}$$

where $d_i(n)$ is the number of partitions of n such that adjacent parts have difference at least 2 and such that the smallest allowed part is i. Again, there are analytic forms for the two identities, respectively:

$$\prod_{j\geq 0} \frac{1}{\left(1 - q^{5j+1}\right)\left(1 - q^{5j+4}\right)} = \sum_{n\geq 0} \frac{q^{n^2}}{\left(1 - q\right)\left(1 - q^2\right)\cdots\left(1 - q^n\right)}, \tag{2.12}$$

$$\prod_{j\geq 0} \frac{1}{\left(1 - q^{5j+2}\right)\left(1 - q^{5j+3}\right)} = \sum_{n\geq 0} \frac{q^{n^2+n}}{\left(1 - q\right)\left(1 - q^2\right)\cdots\left(1 - q^n\right)}. \tag{2.13}$$

It should be noted that, in the cases of both the Rogers-Ramanujan identities and Euler's identity, the algebraic statements were discovered and proved first, and only later were the interpretations in terms of partitions provided.

Broad-sweeping generalizations of the Rogers-Ramanujan identities were provided by Gordon [30], where the sum side conditions featured a new variant. Fix values of k and d. If, for all j, $\lambda_j - \lambda_{j+k} \geq d$, we say that the partition has difference at least d at distance k. Thus, the difference condition in the Rogers-Ramanujan identities is a difference at least 2 at distance 1 condition. The innovation in the sum sides for Gordon's identities is considering sum side conditions where the distance is at least 2:

G : The number of partitions of a non-negative integer into parts not congruent to 0 or $\pm(k - i + 1) \pmod{2k + 1}$ is the same as the number of partitions of n with difference at least 2 at distance $k - 1$ such that 1 appears at most $k - i$ times.

Analogous identities with even moduli were provided by Andrews and Bressoud [?]:

G : The number of partitions of a non-negative integer into parts not congruent to 0 or $\pm(k - i + 1) \pmod{2k}$ is the same as the number of partitions $\pi = (\pi_1, \ldots, \pi_s)$ of n with difference at least 2 at distance $k - 1$ such that 1 appears at most $k - i$ times and, if $\pi_t - \pi_{t+k-2} \leq 1$, then $\pi_t + \pi_{t+1} + \cdots + \pi_{t+k-2} \equiv i + k \pmod{2}$.

Our final example in this survey of partition identities was provided by Stefano Capparelli. Through his study of the affine Lie algebra $A_2^{(2)}$ in his doctoral thesis [16], he discovered (at first conjecturally) a pair of partition identities. The first of these is:

C_1 : The number of partitions of a non-negative integer into parts congruent to ± 2 or ± 3 (mod 12) is the same as the number of partitions of n such that consecutive parts must differ by at least 2, the difference is at least 4, unless consecutive parts add up to a multiple of 3, and 1 is not allowed as a part.

This first of the two identities was initially proven by George Andrews in 1992 [7], and eventually by Capparelli himself using Lie-theoretic means [17]. (For the fascinating story of Andrews's original proof, see [1]).

2.4 Methods

In this section, we will discuss the methodology we used to discover our partition conjectures.

First, we need to discuss the types of conditions that we used (with varying parameters) to generate our sum sides. The following list provides the conditions, together with the corresponding procedures in `IdentityFinder`:

- Smallest part size (`SmPartCheck`): This is the smallest allowable part, along with the maximum multiplicity with which it is allowed to appear. For example, in Gordon's identities, there is a restriction on the number of occurrences of 1 as a part.

- Difference-at-a-distance (`DiffDistCheck`): Fix values of k and d. If, for all j, $\lambda_j - \lambda_{j+k} \geq d$, we say that the partition satisfies the difference d at distance k condition. For example, the two Rogers-Ramanujan identities both feature the difference 2 at distance 1 condition.

- Congruence-at-a-distance (`CapparelliCheck`): If for all j, $\lambda_j \leq \lambda_{j+A} + B$ only if $\lambda_j + \lambda_{j+1} \cdots + \lambda_{j+A}$ is congruent to C (mod D), we say that the partition satisfies the (A, B, C, D)-congruence condition. Using $A = 1$ is an important

specialization. In this case, the condition simplifies to: two consecutive parts differ by at most B only if their sum is congruent to C (mod D). For example, in our notation, the key condition in Capparelli's identities is the $(1, 3, 0, 3)$-congruence condition. The Andrews-Bressoud identities use a congruence condition with $D = 2$. It can be important to consider seemingly "wrong" or "unnatural" congruence conditions (see the "ghost series" in the motivated proof of the Andrews-Bressoud identities in [35]).

The previous three conditions give us eight possible parameters (two for `SmPartCheck`, two for `DiffDistCheck`, and four for `CapparelliCheck`. In practice, we choose small ranges (say, 1 to 5) for each of these parameters (although, in `CapparelliCheck`, C is always chosen to be between 0 and $D-1$, to avoid redundancies). For each combination of parameters, we examine all partitions of n from 0 to N (typically using $N = 30$), using the preceding procedures, and deduce initial terms for the corresponding (infinite) generating function.

We arrive at

$$1 + \sum_{n=1}^{N} b_n q^n,$$

and can use Euler's algorithm to factor the expression just obtained as

$$f(q) = \prod_{m \geq 1} (1 - q^m)^{-a_m}.$$

In practice, we use the implementation of Euler's algorithm in Frank Garvan's `qseries` package [30] as the procedure `prodmake`.

We now consider the sequence $\{a_m\}_{m=1}^{N}$. If this sequence is periodic, with an appropriately small period, then we have a candidate for a new partition identity.

The above process identifies potential candidates for new identities. After potential candidate new identities were identified, we used the On-Line Encyclopedia of Integer Sequences [49] to help determine which ones were already known (for example, in identifying many cases of the Göllnitz-Gordon-Andrews theorem). If the conjectured identity still appeared to be new at this point, we had two options to further verify candidates. Experimentally, we can simply increase the value of N, calculating additional terms

of the sum side, and verifying that this still matches the product side. Alternatively, we can write down the recursions that govern the sum sides, along with their initial conditions. Together, this can allow us to calculate hundreds or thousands of terms on the sum side.

2.5   Results

Naturally, our methods rediscover many known identities. Our initial searches have found six new (conjectured) identities, which were first published in [36]. All six of these conjectures have been verified for at least 1500 terms. Three of them form a single family of mod 9 identities:

I_1 : The number of partitions of a non-negative integer into parts congruent to ± 1 or ± 3 (mod 9) is the same as the number of partitions with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3.

I_2 : The number of partitions of a non-negative integer into parts congruent to ± 2 or ± 3 (mod 9) is the same as the number of partitions with smallest part at least 2 and difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3.

I_3 : The number of partitions of a non-negative integer into parts congruent to ± 3 or ± 4 (mod 9) is the same as the number of partitions with smallest part at least 3 and difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3.

A fourth mod 9 identity appears to be related, but has asymmetric congruence conditions:

I_4 : The number of partitions of a non-negative integer into parts congruent to 2, 3, 5, or 8 (mod 9) is the same as the number of partitions with smallest part at least 2 and difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is congruent to 2 (mod 3).

We also found a pair of mod 12 identities, again with asymmetric congruence conditions:

I_5 : The number of partitions of a non-negative integer into parts congruent to 1, 3, 4, 6, 7, 10, or 11 (mod 12) is the same as the number of partitions with at most one appearance of the part 1 and difference at least 3 at distance 3 such that if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is congruent to 1 (mod 3).

I_6 : The number of partitions of a non-negative integer into parts congruent to 2, 3, 5, 6, 7, 8, or 11 (mod 12) is the same as the number of partitions with smallest part at least 2, at most one appearance of the part 2, and difference at least 3 at distance 3 such that if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is congruent to 2 (mod 3).

Let us illustrate I_1, I_2, and I_3 with an example. There are fourteen partitions of $n = 13$ into parts congruent to ± 1 or ± 3 (mod 9):

13, $10 + 3$, $10 + 1 + 1 + 1$, $8 + 3 + 1 + 1$, $8 + 1 + 1 + 1 + 1 + 1$, $6 + 6 + 1$, $6 + 3 + 3 + 1$, $6 + 3 + 1 + 1 + 1 + 1$, $6 + 1 + 1 + 1 + 1 + 1 + 1 + 1$, $3 + 3 + 3 + 3 + 1$, $3 + 3 + 3 + 1 + 1 + 1 + 1$, $3 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1$, $3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$, and $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$.

There are seven partitions of $n = 13$ into parts congruent to ± 2 or ± 3 (mod 9):

$11 + 2$, $7 + 6$, $7 + 3 + 3$, $7 + 2 + 2 + 2$, $6 + 3 + 2 + 2$, $3 + 3 + 3 + 2 + 2$, and $3 + 2 + 2 + 2 + 2 + 2$.

There are five partitions of $n = 13$ into parts congruent to ± 3 or ± 4 (mod 9):

13, $6 + 4 + 3$, $5 + 5 + 3$, $5 + 4 + 4$, and $4 + 3 + 3 + 3$.

There are fourteen partitions of $n = 13$ that satisfy the sum side conditions of I_1:

13, $10 + 3$, $9 + 4$, $8 + 5$, $7 + 3 + 3$, $11 + 2$, $7 + 4 + 2$, $12 + 1$, $10 + 2 + 1$, $9 + 3 + 1$, $8 + 4 + 1$, $7 + 5 + 1$, $6 + 6 + 1$, and $6 + 4 + 2 + 1$.

This confirms I_1 for $n = 13$. Our somewhat unorthodox ordering of these partitions is to make it apparent that exactly seven of these partitions do not contain 1 as a subpartition, and exactly five of these partitions do not contain 1 or 2 as subpartitions. As the sum side conditions for I_2 and I_3 are exactly the same as for I_1, except that 1

is forbidden as a subpartition and 1 and 2 are forbidden as subpartitions, respectively, this takes care of verifying I_2 and I_3.

Also, to illustrate I_5, there are seventeen partitions of $n = 11$ into parts congruent to 1, 3, 4, 6, 7, 10, or 11 (mod 12):

11, $10+1$, $7+4$, $7+3+1$, $7+1+1+1+1$, $6+4+1$, $6+3+1+1$, $6+1+1+1+1+1$, $4+4+3$, $4+4+1+1+1$, $4+3+3+1$, $4+3+1+1+1+1$, $4+1+1+1+1+1+1+1$, $3+3+3+1+1$, $3+3+1+1+1+1+1$, $3+1+1+1+1+1+1+1+1$, and $1+1+1+1+1+1+1+1+1+1+1$.

To check this, we list the seventeen partitions of $n = 11$ that satisfy the sum side conditions of I_5:

11, $10+1$, $9+2$, $8+3$, $8+2+1$, $7+4$, $7+3+1$, $7+2+2$, $6+5$, $6+4+1$, $6+3+2$, $5+5+1$, $5+4+2$, $5+3+3$, $5+3+2+1$, $4+4+2+1$, and $4+3+3+1$.

2.6 Verification

Using the approaches detailed before, we can verify our conjectures for the first several dozen terms. In order to increase this further, we have to use alternate methods for calculating the sum sides, as detailed in this section. In order to do this, we use generating functions for the sum sides, with added restrictions on the size of the largest part (which means that these will be polynomials). In the limit as the allowable size of the largest part goes to infinity, these polynomials converge to their respective sum sides. This procedure allows us to quickly compute hundreds or thousands of terms for the sum sides, and then verification with the product sides is routine.

This method is similar to one found in [9]. Analogous polynomials have been used in motivated proofs of the Gordon, Göllnitz-Gordon-Andrews, and Andrews-Bressoud identities, respectively (see [42], [23], and [35]).

For every identity, we write recursions for generating functions for partitions satisfying the conditions in the sum side with added upper bound N on the largest part. It is easy to see that these new generating functions are polynomials which agree with the full sum sides up to the coefficient of q^N. From (2.3), it is easy to see that $b_1, \ldots, b_N$

uniquely determine $a_1, \ldots, a_N$.

Let $f_j(q)$ be the generating function for the partitions counted in the sum side of the identity I_j for $j = 1, 2, 3$. Let

$$P_{j,k}(q) = \text{generating function for the partitions counted}$$
$$\text{in the sum side of the identity } I_j, \text{ where } j = 1, 2, 3,$$
$$\text{with the added restriction that the largest part is at most } k.$$

Let $P_{j,k}(q)$ be the generating function for the partitions counted in the sum side of the identity I_j, where $j \in \{1, 2, 3\}$, with the added restriction that the largest part is at most k. It is easy to see that the polynomials $P_{j,k}(q)$ satisfy the same recursion irrespective of the value of j:

It is clear that $P_{j,k}(q)$ is a polynomial, which agrees with $f_j(q)$ up to q^k. Therefore, by Corollary 2.2.2, the product obtained by applying Euler's algorithm to $P_{j,k}(q)$ agrees with the product corresponding to $f_j(q)$ up to the factor $(1 - q^k)^{-a_k}$. It is easy to see that the polynomials $P_{j,k}(q)$ satisfy the same recursion irrespective of the value of j:

$$P_{j,3n} = P_{j,3n-1} + q^{3n} P_{j,3n-2} + q^{3n} q^{3n} P_{j,3n-3} \tag{2.14}$$

$$P_{j,3n+1} = P_{j,3n} + q^{3n+1} P_{j,3n-1} \tag{2.15}$$

$$P_{j,3n+2} = P_{j,3n+1} + q^{3n+2} q^{3n+1} P_{j,3n-1} + q^{3n+2} q^{3n} P_{j,3n-2} + q^{3n+2} P_{j,3n-1}. \tag{2.16}$$

Note that we have presented some exponents in an unsimplified form in our recursions (above and below) to better illustrate how the latter are produced. The initial conditions, however, depend on the value of j:

$$P_{1,1} = 1 + q \qquad P_{1,2} = 1 + q + q^2 + q^3 \qquad P_{1,3} = 1 + q + q^2 + 2q^3 + q^4 + q^6 \tag{2.17}$$

$$P_{2,1} = 1 \qquad P_{2,2} = 1 + q^2 \qquad P_{2,3} = 1 + q^2 + q^3 + q^6 \tag{2.18}$$

$$P_{3,1} = 1 \qquad P_{3,2} = 1 \qquad P_{3,3} = 1 + q^3 + q^6. \tag{2.19}$$

For the remaining mod 9 identity, I_4, letting $Q_k(q)$ be the generating function for

the partitions counted in the sum side of the identity I_4, with the added restriction that the largest part is at most k, the recursions and initial conditions are:

For the remaining mod 9 identity, I_4, letting

$$Q_k(q) = \text{generating function for the partitions counted}$$
$$\text{in the sum side of the identity } I_4,$$
$$\text{with the added restriction that the largest part is at most } k,$$

the recursions are:

$$Q_{3n} = Q_{3n-1} + q^{3n}q^{3n-1}Q_{3n-3} + q^{3n}q^{3n-2}Q_{3n-4} + q^{3n}Q_{3n-3} \tag{2.20}$$

$$Q_{3n+1} = Q_{3n} + q^{3n+1}q^{3n+1}Q_{3n-2} + q^{3n+1}Q_{3n-1} \tag{2.21}$$

$$Q_{3n+2} = Q_{3n+1} + q^{3n+2}Q_{3n}, \tag{2.22}$$

with the initial conditions:

$$Q_0 = 1, \ Q_1 = 1, \ Q_2 = 1 + q^2, \ Q_3 = 1 + q^2 + q^3 + q^5.$$

For the first mod 12 identity, I_5, let $R_{n,a}(q)$ be the generating function of partitions with largest part at most n and at most a parts equalling n, in addition to the given constraints in the sum-sides. The following recursions and initial conditions are satisfied by these polynomials:

Now, let us turn to the mod 12 identities. For the first mod 12 identity, I_5, let

$$R_{n,a}(q) = \text{generating function of partitions with largest part at most } n$$
$$\text{and at most } a \text{ parts equalling } n$$
$$\text{in addition to the given constraints in the sum-sides.} \tag{2.23}$$

It is easy to see that we need only consider $a \in \{1, 2\}$. The following recursions and initial conditions are satisfied by these polynomials:

$$R_{n,1} = q^n R_{n-1,2} - q^n q^{n-1} q^{n-2} q^{n-2} R_{n-4,1} + R_{n-1,2} \tag{2.24}$$

$$R_{n,2} = q^n q^n R_{n-2,1} + R_{n,1} \tag{2.25}$$

$$R_{1,1} = 1 + q \tag{2.26}$$

$$R_{1,2} = 1 + q \tag{2.27}$$

$$R_{2,1} = 1 + q + q^2 + q^3 \tag{2.28}$$

$$R_{2,2} = 1 + q + q^2 + q^3 + q^4 \tag{2.29}$$

$$R_{3,1} = 1 + q + q^2 + 2q^3 + 2q^4 + q^5 + q^6 + q^7 \tag{2.30}$$

$$R_{3,2} = 1 + q + q^2 + 2q^3 + 2q^4 + q^5 + 2q^6 + 2q^7 \tag{2.31}$$

$$R_{4,1} = 1 + q + q^2 + 2q^3 + 3q^4 + 2q^5 + 3q^6 + 4q^7 + 2q^8 + q^9 + 2q^{10} + q^{11} \tag{2.32}$$

$$R_{4,2} = 1 + q + q^2 + 2q^3 + 3q^4 + 2q^5 + 3q^6 + 4q^7 + 3q^8 + 2q^9 + 3q^{10} + 2q^{11}. \tag{2.33}$$

Finally, defining $S_{n,a}(q)$ analogously for I_6, we compute

$$S_{n,1} = q^n S_{n-1,1} + S_{n-1,2} \tag{2.34}$$

$$S_{n,2} = q^n q^n q^{n-1} S_{n-3,2} + q^n q^n S_{n-2,1} + S_{n,1} \tag{2.35}$$

$$S_{1,1} = 1 \tag{2.36}$$

$$S_{1,2} = 1 \tag{2.37}$$

$$S_{2,1} = 1 + q^2 \tag{2.38}$$

$$S_{2,2} = 1 + q^2 \tag{2.39}$$

$$S_{3,1} = 1 + q^2 + q^3 + q^5 \tag{2.40}$$

$$S_{3,2} = 1 + q^2 + q^3 + q^5 + q^6 + q^8. \tag{2.41}$$

Practically speaking, if we only desire to verify the conjectures up to some $n = N$, we can compute all of the polynomials above modulo q^{N+1}. This significantly speeds up the calculations.

2.7 Future work

The results of this chapter provide many ideas for future work. First, it is always possible to expand the parameter-space search by incorporating more innovative conditions on the sum sides (for instance, the Göllnitz-Gordon-Andrews identities [10], the Andrews-Santos identities [12], etc.) or by optimizing the currently existing code to search for more possible combinations of parameter values. We hope that more identities could be found in this way. It would be interesting to examine the sum sides of more recent partition identities to see if they can inspire additional checks to build in to the package. For one intriguing example, see the new difference conditions on the sum sides of the conjectures in Nandi [48].

Recent research (see, for instance, [20]) has focused on providing overpartition analogues of many classical partition identities. Overpartitions are similar to partitions, except that you are allowed to overline — that is, mark in some special way — the final occurrence of any part in the partition. It would be interesting to extend our methods to consider overpartitions and, more generally, multi-color partition identities. However, one main challenge is that there are many more overpartitions than partitions of a given integer. Using a naïve approach, it may be difficult to calculate out enough terms to form reasonable conjectures.

Obviously, it would be nice to be able to prove any or all of the above conjectured identities, but this has proven to be difficult so far. We now discuss some possible means of attack.

As observed by George Andrews [5], we can rewrite the generating function for the

product side of I_4 as

$$\prod_{j \geq 1} \frac{1}{\left(1 - q^{9j-7}\right)\left(1 - q^{9j-6}\right)\left(1 - q^{9j-4}\right)\left(1 - q^{9j-1}\right)}$$

$$= \prod_{j \geq 1} \frac{\left(1 - q^{9j-3}\right)\left(1 - q^{9j}\right)}{\left(\left(1 - q^{9j-7}\right)\left(1 - q^{9j-4}\right)\left(1 - q^{9j-1}\right)\right)\left(\left(1 - q^{9j-6}\right)\left(1 - q^{9j-3}\right)\left(1 - q^{9j}\right)\right)}$$

$$= \prod_{j \geq 1} \frac{\left(1 - q^{9j-3}\right)\left(1 - q^{9j}\right)}{\left(1 - q^{3j-1}\right)\left(1 - q^{3j}\right)}$$

$$= \prod_{j \geq 1} \left(1 + q^{3j-1} + q^{6j-2}\right)\left(1 + q^{3j} + q^{6j}\right),$$

which is the generating function for partitions into parts congruent to 0 or 2 (mod 3), where each part is allowed to appear at most twice.

This sort of reformulation of the product side can be done in many cases, such as Schur's theorem and Capparelli's theorems. The analogous process was a key step in the first proof of Capparelli's first identity in [9]. Andrews's work in [10] was influenced by a desire to explore this more closely. Unfortunately, no similar reformulations of the generating functions for the product sides of I_5—I_{6a} have been found.

It would also greatly assist in the proofs of any of these identities if an analytic sum side could be discovered.

A different possible approach may be through a "motivated proof" framework (see [11], [42], [23], [35], or the chapter dealing with motivated proofs). The first three identities, I_1, I_2, and I_3, all appear to be connected with the affine Lie algebra $D_4^{(3)}$, and preliminary investigations into a motivated proof using the Macdonald identity corresponding to $D_4^{(3)}$ have begun. While all of the previous papers used the Jacobi Triple Product identity, Coulson [22] has made significant progress on a proof of the Rogers-Ramanujan identities, but using the Macdonald identity for $A_2^{(1)}$. This is a major breakthrough, as this Macdonald identity gives a double sum instead of a single sum. So, there is hope that a motivated proof using the Macdonald identity corresponding to $D_4^{(3)}$ is possible. However, this connection only exists if the product side is symmetric. Hence, we should not expect Lie-theoretic interpretations of the remaining conjectures.

Perhaps more in the spirit of this chapter is to consider other possible sum sides, especially by looking at other already established partition identities, such as the Göllnitz-Gordon-Andrews identities.

¡Rewrite¿ As presented in Chapter 7 of [6] (which also gives the Rogers-Ramanujan and Gordon-Andrews identities), these identities state that for any $k \geq 2$ (for $k = 1$, one gets the trivial identity $1 = 1$) and $i = 1, \ldots, k$,

$$\prod_{\substack{m \geq 1, \, m \not\equiv 2 \,(\mathrm{mod}\,4), \\ m \not\equiv 0, \, 2k \pm (2i-1) \,(\mathrm{mod}\,4k)}} \frac{1}{1 - q^m} = \sum_{n \geq 0} d_{k,i}(n) q^n,$$

where $d_{k,i}(n)$ is the number of partitions $(b_1, \ldots, b_s)$ of n (with $b_p \geq b_{p+1}$), satisfying the following conditions:

1. No odd parts are repeated,

2. $b_p - b_{p+k-1} \geq 2$ if b_p is odd,

3. $b_p - b_{p+k-1} > 2$ if b_p is even, and

4. at most $k - i$ parts are equal to 1 or 2.

Alladi and Andrews [2]: The number of partitions of n into distinct parts congruent to 3, 4, 5, or 8 (mod 8) is equal to the number of partitions of n into distinct parts > 1 which are $\not\equiv 2$ (mod 4) such that the difference between consecutive parts is > 4 unless they are both multiples of 4 or they add up to a multiple of 8.

Note that here, for the first time, we specifically forbid certain congruence classes in the sum side.

(Alternatively, this can be viewed as a special case of congruence-at-a-distance (`CapparelliCheck`) at distance 0.)

Chapter 3

More on partition identities

Following the success of the `IdentityFinder` project [36], as discussed in the previous chapter, I turned my attention towards discovering additional (conjectured) identities. One major issue with the preceding work is that it assumes a very specific type of initial conditions. However, there is no reason that we should expect all partition identities to have this form. At this point, we will discuss a few more previously discovered identities, which will prove helpful in guiding our searches.

Capparelli actually conjectured two identities in his thesis [16]. The second, not as widely known as his first, states:

C_2 : The number of partitions of a non-negative integer into distinct parts congruent to 0, 1, 3, or 5 (mod 12) is the same as the number of partitions of n such that consecutive parts must differ by at least 2, the difference is at least 4, unless consecutive parts add up to a multiple of 3, and 2 is not allowed as a part.

(Note that the product side is written in terms of partitions into distinct parts contained in certain residue classes. It is quite easy to show that Capparelli's first identity has a similar formulation using partitions into distinct parts.)

Another pair to consider are Göllnitz's (big) theorem [29], together with its dual (found by Alladi and Andrews [3]):

G : The number of partitions of a non-negative integer into parts congruent to 2, 5, or 11 (mod 12) is the same as the number of partitions such that consecutive parts must differ by at least 6, the difference is at least 7, unless the parts are congruent to 2, 4, or 5 (mod 6), and 1 and 3 are not allowed to appear in the partition.

G' : The number of partitions of a non-negative integer into parts congruent to 1, 7, or

10 (mod 12) is the same as the number of partitions such that consecutive parts must differ by at least 6, the difference is at least 7, unless the parts are congruent to 1, 2, or 5 (mod 6), with the exception that $6 + 1$ may appear in the partition.

These are dual identities: the allowable residue classes on the product side for Göllnitz's theorem, 2, 5, and 11 (mod 12), are exactly the negatives of the residue classes for the dual (-2, -5, and -11 (mod 12)). A similar phenomenon happens with the congruence classes in the sum sides. However, the initial conditions have significantly changed.

Unfortunately, none of these identities are of the form required using the methods of the previous chapter. In particular, our procedure for checking initial conditions, `SmPartCheck`, did not permit the sort of initial conditions that appear in these three identities. For example, using `SmPartCheck`, there is no way to forbid 1 and 3 from appearing in our partitions, while simultaneously permitting 2. Moreover, in Alladi and Andrews's theorem, the initial condition is a subpartition that would otherwise be forbidden, but is explicitly allowed.

In this vein, I reworked part of `IdentityFinder` to incorporate a more general set of initial conditions. In particular, I created a list of partitions that were somewhat likely to appear in initial conditions. The initial conditions I was considering then became forbidding some small subset of these partitions from appearing in our partitions. This improvement led to the discovery of three new conjectures, which are listed below as I_{4a}, I_{5a}, and I_{6a}. Additionally, I_4, I_5, and I_6 are repeated below, for purposes of comparison.

I_4 : The number of partitions of a non-negative integer into parts congruent to 2, 3, 5, or 8 (mod 9) is the same as the number of partitions with difference at least 3 at distance 2 such that, if two consecutive parts differ by at most 1, then their sum is congruent to 2 (mod 3), and 1 is not allowed to appear in the partition.

I_{4a} : The number of partitions of a non-negative integer into parts congruent to 1, 4, 6, or 7 (mod 9) is the same as the number of partitions with difference at least 3 at distance 2 such that, if two consecutive parts differ by at most 1, then their sum

is congruent to 1 (mod 3), and $2 + 2$ is not allowed to appear in the partition.

I_5 : The number of partitions of a non-negative integer into parts congruent to 1, 3, 4, 6, 7, 10, or 11 (mod 12) is the same as the number of partitions with difference at least 3 at distance 3 such that, if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is congruent to 1 (mod 3), and $1 + 1$ is not allowed to appear in the partition.

I_{5a} : The number of partitions of a non-negative integer into parts congruent to 1, 2, 5, 6, 8, 9, or 11 (mod 12) is the same as the number of partitions with difference at least 3 at distance 3 such that, if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is congruent to 2 (mod 3), and $2 + 2 + 1$ is not allowed to appear in the partition.

I_6 : The number of partitions of a non-negative integer into parts congruent to 2, 3, 5, 6, 7, 8, or 11 (mod 12) is the same as the number of partitions with difference at least 3 at distance 3 such that, if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is congruent to 2 (mod 3), 1 is not allowed to appear in the partition, and $2 + 2$ is not allowed to appear in the partition.

I_{6a} : The number of partitions of a non-negative integer into parts congruent to 1, 4, 5, 6, 7, 9, or 10 (mod 12) is the same as the number of partitions with difference at least 3 at distance 3 such that, if parts at distance two differ by at most 1, then their sum (together with the intermediate part) is congruent to 1 (mod 3), and 2 is not allowed to appear in the partition.

Note that I_{4a}, I_{5a}, and I_{6a} are the duals of I_4, I_5, and I_6, respectively. For example, if we negate the allowable residue classes for I_5, we obtain $-1, -3, -4, -6, -7, -10$, and -11 (mod 12), which we rewrite as 1, 2, 5, 6, 8, 9, and 11 (mod 12) — which are exactly the congruence classes for I_{5a}.

Additionally, the sum side conditions are also dual: the requirement that the sum is congruent to 1 (mod 3) in I_5 is now a requirement that the sum is congruent to 2

(mod 3) in I_{5a}.

Furthermore, note that the iteration of `IdentityFinder` used in the work done in the previous chapter would not have been able to find I_{5a}, as the initial condition states only that $2 + 2 + 1$ is not allowed to appear in the partition.

The above three comments hold true (with appropriate modifications) for each of I_{4a}, I_{5a}, and I_{6a}. The hope is that finding these dual conjectures will help guide a proof of them (and it is likely that each conjecture will be proved simultaneously with its dual), but proofs have remained elusive. As with I_4, a similar calculation demonstrates that the generating function for the product side of I_{4a} equals the generating function for partitions into parts congruent to 0 or 1 (mod 3), where each part is allowed to appear at most twice. So far, experimentation with the type of initial conditions where certain subpartitions are explicitly permitted (as in Alladi and Andrews's dual of Göllnitz's theorem) have not produced any new conjectures.

Chapter 4

Experimentation, motivated proofs, and ghost series

Another way that I have used experimental mathematics to study partition idenities was in my project with Kanade, Lepowsky, and Sills to provide a "motivated proof" of the Andrews-Bressoud identities. The full story is told in [35], in which all of the calculations are rigorously proven. This chapter chronicles the computer experimentation I contributed that aided the project.

The Andrews-Bressoud identities state that for any $k \geq 2$ and $i \in \{1, \ldots, k\}$, the number of partitions of a non-negative integer n into parts not congruent to 0 or $\pm (k - i + 1) \pmod{2k}$ is the same as the number of partitions λ of n such that the partition has difference at least 2 at distance $k - 1$ (that is, $\lambda_t - \lambda_{t+k-1} \geq 2$), $\lambda_t - \lambda_{t+k-2} \leq 1$ only if $\lambda_t + \lambda_{t+1} + \cdots + \lambda_{t+k-2} \equiv i + k \pmod{2}$, and at most $k - i$ parts are equal to 1. (Note that we have replaced r by $k - i + 1$ in the statement of the main theorem of [?].)

The goal of the project was to provide a motivated proof of these identities. The first motivated proof was provided by Andrews and Baxter, who proved the Rogers-Ramanujan identities using this framework [11]. The basic idea of motivated proofs is as follows. Beginning with the product sides written as formal power series, one constructs a sequence of new formal power series as linear combinations of the previous terms in the sequence. These are all proven to be of the form $1 + q^j + \cdots$ for ever-increasing values of j (this observation is called the Empirical Hypothesis). Then, this fact is combined with the recursions that are used to generate the sequence of formal power series, which are observed to be the same as the recursions for the sum sides of the identities, completing the proof. As compared to traditional proofs of partition identities, motivated proofs may provide some additional insight into the truth of the

identities. Motivated proofs also fit in very nicely with the Lie-theoretic approach mentioned in section —.

Subsequently, Lepowsky and Zhu provided a motivated proof of the Gordon identities [42], and Coulson, Kanade, Lepowsky, McRae, Qi, Sadowski, and myself did the same for the Göllnitz-Gordon-Andrews identities [23]. Recently, Chris Sadowski, Collin Takita, and I have completed a motivated proof of an overpartitition identity [55]. However, the work behind the motivated proof described in this section is different than some of the others. In particular, our "Empirical Hypothesis" was truly empirical, which separates it from [11], [42] and [23].

Here is where the experimental component of the project came into play. It is easy to write down and calculate the generating functions for the product sides of the Andrews-Bressoud identities:

$$\frac{\prod_{m \geq 1}(1 - q^{2km})(1 - q^{2km-k-i+1})(1 - q^{2km-k+i-1})}{\prod_{m \geq 1}(1 - q^m)}$$

Fixing some particular value of k — say, $k = 5$ — we calculated the initial terms of B_1, B_2, B_3, B_4, and B_5:

$$B_i = \sum_{n \geq 0} b_i(n) q^n,$$

where $b_i(n)$ is the number of partitions $\lambda\lambda = (\lambda_1, \ldots, \lambda_s)$ of n (with $\lambda_t \geq \lambda_{t+1}$) such that the partition has difference at least 2 at distance 4 (that is, $\lambda_t - \lambda_{t+4} \geq 2$), $\lambda_t - \lambda_{t+3} \leq 1$ only if $\lambda_t + \lambda_{t+1} + \cdots + \lambda_{t+3} \equiv i + 5 \pmod 2$, and at most $5 - i$ parts are equal to 1.

Initial explorations to create higher shelves merely using these series proved unfruitful. The idea I had was to consider the series with the "wrong" parity conditions, which we called "ghost series". Independently of this work, the ghost series (without the name) from the zeroth shelf were discussed by K. Kurşungöz in [37]. They can be recovered by setting $d = 2$, $s = 1$ in $_dB_{k,a}^s(n)$ in [37].

Accordingly, we then introduced series $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{B}_3$, $\tilde{B}_4$, and $\tilde{B}_5$, where

$$\tilde{B}_i = \sum_{n \geq 0} \tilde{b}_i(n) q^n,$$

where $\tilde{b}_i(n)$ is the number of partitions $\lambda\lambda = (\lambda_1, \ldots, \lambda_s)$ of n (with $\lambda_t \geq \lambda_{t+1}$) such that the partition has difference at least 2 at distance 4 (that is, $\lambda_t - \lambda_{t+4} \geq 2$), $\lambda_t - \lambda_{t+3} \leq 1$

only if $\lambda_t + \lambda_{t+1} + \cdots + \lambda_{t+3} \equiv i + 6 \pmod{2}$, and at most $5 - i$ parts are equal to 1. Note that the definition here is nearly the same as the corresponding one for B_r. The only difference is that the parity condition has been changed.

Following the program developed in [42], it was quite clear that the most natural way to construct the sum sides for the higher shelves was as follows: for $r = (k-1)J + i$, with $i \in \{1, \dots, k\}$,

$$B_r = \sum_{n \geq 0} b_{k,r}(n) q^n,$$

where $b_{k,r}(n)$ denotes the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ of n such that:

- $\lambda_t - \lambda_{t+k-1} \geq 2$,

- $\lambda_{\ell(\lambda)} \geq J + 1$,

- $m_{J+1}(\lambda) \leq k - i$,

- $\lambda_t - \lambda_{t+k-2} \leq 1$ only if $\lambda_t + \lambda_{t+1} + \cdots + \lambda_{t+k-2} \equiv r + k \equiv (k-1)J + i + k \pmod{2}$.

Note that setting $J = 0$ in the above work allows us to recover the Andrews-Bressoud sum sides.

Then, our ghost sum sides will be: for $r = (k-1)J + i$, with $i \in \{2, \dots, k\}$,

$$\tilde{B}_r = \sum_{n \geq 0} \tilde{b}_{k,r}(n) q^n,$$

where $\tilde{b}_{k,r}(n)$ denotes the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ of n such that:

1. $\lambda_t - \lambda_{t+k-1} \geq 2$,

2. $\lambda_{\ell(\lambda)} \geq J + 1$,

3. $m_{J+1}(\lambda) \leq k - i$,

4. $\lambda_t - \lambda_{t+k-2} \leq 1$ only if $\lambda_t + \lambda_{t+1} + \cdots + \lambda_{t+k-2} \equiv r + k + 1 \equiv (k-1)J + i + k + 1$ (mod 2).

Using Maple, I then calculated out approximately the first 40 terms of each sum side, and observed that the following relations held:

$$B_6 = \tilde{B}_5$$
$$= \left(B_4 - \tilde{B}_5 \right) / q$$
$$B_7 = \left(\tilde{B}_4 - B_5 \right) / q$$
$$= \left(B_3 - \tilde{B}_4 \right) / q^2$$
$$B_8 = \left(\tilde{B}_3 - B_4 \right) / q^2$$
$$= \left(B_2 - \tilde{B}_3 \right) / q^3$$
$$B_9 = \left(\tilde{B}_2 - B_3 \right) / q^3$$
$$= \left(B_1 - \tilde{B}_2 \right) / q^3$$

$$B_{10} = \tilde{B}_9$$
$$= \left(B_8 - \tilde{B}_9 \right) / q^2$$
$$B_{11} = \left(\tilde{B}_8 - B_9 \right) / q^2$$
$$= \left(B_7 - \tilde{B}_8 \right) / q^4$$
$$B_{12} = \left(\tilde{B}_7 - B_8 \right) / q^4$$
$$= \left(B_6 - \tilde{B}_7 \right) / q^6$$
$$B_{13} = \left(\tilde{B}_6 - B_7 \right) / q^6$$
$$= \left(B_5 - \tilde{B}_6 \right) / q^8$$

Furthermore, it was easy to observe the appropriate Empirical Hypotheses here:

$$B_5 = 1 + q^2 + \cdots \qquad \qquad \tilde{B}_5 = 1 + q^2 + \cdots$$
$$B_6 = 1 + q^2 + \cdots \qquad \qquad \tilde{B}_6 = 1 + q^2 + \cdots$$
$$B_7 = 1 + q^2 + \cdots \qquad \qquad \tilde{B}_7 = 1 + q^2 + \cdots$$
$$B_8 = 1 + q^2 + \cdots \qquad \qquad \tilde{B}_8 = 1 + q^2 + \cdots$$

$$B_9 = 1 + q^3 + \cdots \qquad\qquad \tilde{B}_9 = 1 + q^3 + \cdots$$

$$B_{10} = 1 + q^3 + \cdots \qquad\qquad \tilde{B}_{10} = 1 + q^3 + \cdots$$

$$B_{11} = 1 + q^3 + \cdots \qquad\qquad \tilde{B}_{11} = 1 + q^2 + \cdots$$

$$B_{12} = 1 + q^3 + \cdots \qquad\qquad \tilde{B}_{12} = 1 + q^3 + \cdots$$

$$B_{13} = 1 + q^4 + \cdots \qquad\qquad \tilde{B}_{13} = 1 + q^4 + \cdots$$

$$B_{14} = 1 + q^4 + \cdots \qquad\qquad \tilde{B}_{14} = 1 + q^4 + \cdots$$

$$B_{15} = 1 + q^4 + \cdots \qquad\qquad \tilde{B}_{15} = 1 + q^4 + \cdots$$

$$B_{16} = 1 + q^4 + \cdots \qquad\qquad \tilde{B}_{16} = 1 + q^4 + \cdots$$

At this point, the above evidence demonstrated that these "ghost series" were the best way to attack the motivated proof, as shown in [35].

Chapter 5

Experimental mathematics and the Laurent phenomenon

5.1 Introduction

Let K_r (the Kontsevich map) be the automorphism of a noncommutative plane defined by

$$K_r : (x, y) \mapsto \left(xyx^{-1}, (1 + y^r) \, x^{-1} \right).$$

Maxim Kontsevich conjectured that, for any $r_1, r_2 \in \mathbb{N}$, the iterates

$$\ldots K_{r_2} K_{r_1} K_{r_2} K_{r_1} (x, y)$$

are all given by noncommutative Laurent polynomials in x and y. This is known as the Laurent phenomenon. The conjecture was proved in special cases for certain values of r_1 and r_2 (see [57], [58], [25], and [26]), sometimes also with the positivity conjecture (that all of the Laurent polynomials have nonnegative integer coefficients), and sometimes replacing $1 + y^r$ with any monic palindromic polynomial. Eventually, Berenstein and Retakh [15] gave an elementary proof of the Kontsevich conjecture for general r_1 and r_2, while Rupel [51] subsequently proved it using the Lee-Schiffler Dyck path model (see [43]) while also settling the positivity conjecture.

Later, Berenstein and Retakh [14] extended their methods to consider a more general class of recurrences given by $Y_{k+1} Y_{k-1} = h_k \left(a_{k-1,k} Y_k a_{k,k+1} \right)$, where $h_k \in \mathbb{Q}[x]$ and $h_k(x) = h_{k-2}(x)$ for all $k \in \mathbb{Z}$, $Y_1 a_{12} Y_2 a_{23} = a_{32} Y_2 a_{21} Y_1$, and $a_{k,k\pm1}$ are defined recursively by $a_{k+2,k+1} = a_{k-1,k}^{-1}$ and $a_{k+1,k+2} = a_{k,k+1}^{-1}$. Proceeding in a similar fashion to their previous paper, they prove the Laurent phenomenon for these recurrences where $h_k = 1 + x^{r_k}$.

In the following sections, we endeavor to expand the methods of Berenstein and Retakh [14] to higher-order recurrences, and to using monic palindromic polynomials

instead of $1 + x^r$. In addition to proving our results, we provide a computer program that implements and verifies the formulas found in these sections.

5.2 Preliminaries

Following section 4 of Berenstein and Retakh [14], let $K \geq 2$. Let $\mathcal{F}_K$ denote the $\mathbb{Q}$-algebra generated by $a_1^{\pm 1}$, $a_2^{\pm 1}$, ..., $a_K^{\pm 1}$, $b_1^{\pm 1}$, $b_2^{\pm 1}$, ..., $b_K^{\pm 1}$, and let $\mathcal{F}_K(Y_1, \ldots, Y_K)$ denote the algebra generated by $\mathcal{F}_K$ and $Y_1, \ldots, Y_K$, subject to the relation

$$Y_1 a_1 Y_2 a_2 \cdots Y_K a_K = b_K Y_K b_{K-1} Y_{K-1} \cdots b_1 Y_1. \tag{5.1}$$

For $n \in \mathbb{Z}$, define a_n and b_n recursively by

$$a_{n+K} = b_n^{-1} \tag{5.2}$$

$$b_{n+K} = a_n^{-1}. \tag{5.3}$$

Suppose we have a sequence of monic palindromic polynomials $h_n \in \mathbb{Q}[x]$ such that $h_n = h_{n-K}$ for all $n \in \mathbb{Z}$. Let us write $h_n(x) = \sum_{i=0}^{d_n} P_{n,i} x^i$, so $P_{n,0} = P_{n,d_n} = 1$ for all n. Recursively define $Y_n \in \mathcal{F}_K(Y_1, \ldots, Y_K)$ for $n \in \mathbb{Z} \setminus \{1, \ldots, K\}$ by

$$Y_{n+K} Y_n = h_n \left(a_n Y_{n+1} a_{n+1} Y_{n+2} \cdots Y_{n+K-1} a_{n+K-1} \right). \tag{5.4}$$

Define

$$Y_{n,m}^- = a_{n-1} Y_n a_n Y_{n+1} \cdots a_{m-1} Y_m a_m \qquad\qquad n \leq m$$

$$Y_{n,m}^+ = b_n Y_n b_{n-1} Y_{n-1} \cdots b_m Y_m b_{m-1} \qquad\qquad n \geq m,$$

while also defining $Y_{n,n-1}^- = a_{n-1}$ and $Y_{n,n+1}^+ = b_n$. Then, (5.4) becomes

$$Y_{n+K} Y_n = h_n \left(Y_{n+1,n+K-1}^- \right). \tag{5.5}$$

Proposition 5.2.1. For all $n \in \mathbb{Z}$, we also have

$$Y_n Y_{n+K} = h_n \left(Y_{n+K-1,n+1}^+ \right) \tag{5.6}$$

$$Y_n Y_{n+1,n+K-1}^- = Y_{n+K-1,n+1}^+ Y_n. \tag{5.7}$$

Proof. We proceed by induction on n. (We will only prove these for $n \geq 1$; the proof for $n < 1$ is similar.) Note that (5.1) gives us the base case of (5.7) for $n = 1$. Now, suppose that (5.7) holds for some $n \geq 1$. Conjugating (5.5) on the left by Y_n gives

$$Y_n Y_{n+K} = Y_n h_n \left(Y_{n+1,n+K-1}^- \right) Y_n^{-1},$$

but, since

$$Y_n \left(Y_{n+1,n+K-1}^- \right)^i Y_n^{-1} = \left(Y_{n+K-1,n+1}^+ \right)^i Y_n Y_n^{-1} = \left(Y_{n+K-1,n+1}^+ \right)^i$$

for all $i \geq 0$ (by the inductive hypothesis), we conclude that

$$Y_n Y_{n+K} = h_n \left(Y_{n+K-1,n+1}^+ \right),$$

which is (5.6).

Now, we desire to prove (5.7) for $n+1$. This is equivalent to proving $Y_{n+1,n+K-1}^- Y_{n+K} = Y_{n+K} Y_{n+K-1,n+1}^+$ (multiply each side by $a_n^{-1} = b_{n+K}$ on the left and $a_{n+K} = b_n^{-1}$ on the right to recover the original expression). We calculate

$$
\begin{aligned}
Y_{n+1,n+K-1}^- Y_{n+K} &= Y_n^{-1} Y_{n+K-1,n+1}^+ Y_n Y_{n+K} \\
&= Y_n^{-1} Y_{n+K-1,n+1}^+ h_n \left(Y_{n+K-1,n+1}^+ \right) \\
&= Y_n^{-1} h_n \left(Y_{n+K-1,n+1}^+ \right) Y_{n+K-1,n+1}^+ \\
&= Y_n^{-1} Y_n Y_{n+K} Y_{n+K-1,n+1}^+ \\
&= Y_{n+K} Y_{n+K-1,n+1}^+,
\end{aligned}
$$

using (5.7) for n (the inductive hypothesis) in the first equality and (5.6) in the second and fourth equalities. $\qquad\square$

Lemma 5.2.2. For $s \geq 0$, we have $\left(Y_{1,K-1}^- \right)^s Y_K = Y_K \left(Y_{K-1,1}^+ \right)^s$.

Proof. Using (5.1) and (5.2), we note that

$$
\begin{aligned}
Y_{1,K-1}^{-}Y_K &= a_0 Y_1 a_1 \cdots Y_{K-1} a_{K-1} Y_K \\
&= a_0 \left(Y_1 a_1 \cdots Y_{K-1} a_{K-1} Y_K a_K \right) a_K^{-1} \\
&= b_K^{-1} \left(b_K Y_K b_{K-1} Y_{K-1} \cdots b_1 Y_1 \right) b_0 \\
&= Y_K b_{K-1} Y_{K-1} \cdots b_1 Y_1 b_0 \\
&= Y_K Y_{K-1,1}^{+}.
\end{aligned}
$$

The general claim follows by induction. $\qquad\square$

5.3 Results

Let $\mathcal{A}_n$ be the subalgebra of $\mathcal{F}_K (Y_1, \ldots, Y_K)$ generated by $\mathcal{F}_K$ and $Y_n, \ldots, Y_{n+2K-1}$. We now state our main result.

Theorem 5.3.1. For all $n \in \mathbb{Z}$, $\mathcal{A}_n = \mathcal{A}_0$.

Proof. It is enough to show that $\mathcal{A}_{n+1} = \mathcal{A}_n$. Without loss of generality, we let $n = 0$. So, we try to show that $Y_{2K} \in \mathcal{A}_0$. By (5.6) and the definition of $h_K(x)$, we find

$$
\begin{aligned}
Y_{2K} &= Y_K^{-1} h_K \left(Y_{2K-1,K+1}^{+} \right) \\
&= Y_K^{-1} \sum_{i=0}^{d_K-1} P_{K,i} \left(Y_{2K-1,K+1}^{+} \right)^i + Y_K^{-1} \left(Y_{2K-1,K+1}^{+} \right)^{d_K}, \tag{5.8}
\end{aligned}
$$

as h_K is a monic palindromic polynomial.

We would like to find an expression for $Y_K^{-1} \left(Y_{2K-1,K+1}^+ \right)^{d_K}$. Using (5.4), we calculate

$$Y_0 = Y_K^{-1} \left(1 + \sum_{m=1}^{d_0} P_{0,m} \left(Y_{1,K-1}^- \right)^m \right)$$

$$Y_K^{-1} = Y_0 - Y_K^{-1} \sum_{m=1}^{d_0} P_{0,m} \left(Y_{1,K-1}^- \right)^m$$

$$Y_K^{-1} \left(Y_{2K-1,K+1}^+ \right)^{d_K} = Y_0 \left(Y_{2K-1,K+1}^+ \right)^{d_K} - Y_K^{-1} \sum_{m=1}^{d_0} P_{0,m} \left(Y_{1,K-1}^- \right)^m \left(Y_{2K-1,K+1}^+ \right)^{d_K}$$

$$= Y_0 \left(Y_{2K-1,K+1}^+ \right)^{d_K} - Y_K^{-1} \sum_{m=1}^{d_0} P_{0,m} \left(Y_{1,K-1}^- \right)^m \left(Y_{2K-1,K+1}^+ \right)^m \left(Y_{2K-1,K+1}^+ \right)^{d_K - m}.$$

$$(5.9)$$

$$- Y_K^{-1} \sum_{m=1}^{d_0} P_{0,m} \left(Y_{1,K-1}^- \right)^m \left(Y_{2K-1,K+1}^+ \right)^m \left(Y_{2K-1,K+1}^+ \right)^{d_K - m}.$$

$$(5.10)$$

Now, we would like a formula for $\left(Y_{1,K-1}^- \right)^m \left(Y_{2K-1,K+1}^+ \right)^m$. Define $h_n^\downarrow (x) = \sum_{i=1}^{d_n} P_{n,i} x^i = h_n(x) - 1$ and $h_n^{\downarrow\downarrow}(x) = \sum_{i=1}^{d_n} P_{n,i} x^{i-1} = \frac{h_n(x) - 1}{x}$.

Lemma 5.3.2. For $m \geq 0$, we have

$$\left(Y_{1,K-1}^- \right)^m \left(Y_{2K-1,K+1}^+ \right)^m = 1 + \sum_{s=0}^{m-1} \left(Y_{1,K-1}^- \right)^s \left(\sum_{j=1}^{K-1} Y_{1,j-1}^- h_j^\downarrow \left(Y_{K+j-1,j+1}^+ \right) Y_{K+j-1,K+1}^+ \right) \left(Y_{2K-1,K+1}^+ \right)^s$$

Proof. We proceed by induction on m. The case $m = 0$ is trivial. For the case $m = 1$, we calculate $Y_{1,K-1}^- Y_{2K-1,K+1}^+$.

Lemma 5.3.3. For $0 \leq l$,

$$Y_{1,l}^- Y_{K+l,K+1}^+ = 1 + \sum_{j=1}^{l} Y_{1,j-1}^- h_j^\downarrow \left(Y_{K+j-1,j+1}^+ \right) Y_{K+j-1,K+1}^+$$

Proof. The base case $l = 0$ simply reduces to $a_0 b_K = 1$, which checks. Now, assuming

the inductive hypothesis,

$$\begin{aligned}
Y_{1,l+1}^{-}Y_{K+l+1,K+1}^{+} &= Y_{1,l}^{-}Y_{l+1}a_{l+1}b_{K+l+1}Y_{K+l+1}Y_{K+l,K+1}^{+} \\
&= Y_{1,l}^{-}Y_{l+1}Y_{K+l+1}Y_{K+l,K+1}^{+} \\
&= Y_{1,l}^{-}h_{l+1}\left(Y_{K+l,l+2}^{+}\right)Y_{K+l,K+1}^{+} \\
&= Y_{1,l}^{-}h_{l+1}^{\downarrow}\left(Y_{K+l,l+2}^{+}\right)Y_{K+l,K+1}^{+} + Y_{1,l}^{-}Y_{K+l,K+1}^{+} \\
&= Y_{1,l}^{-}h_{l+1}^{\downarrow}\left(Y_{K+l,l+2}^{+}\right)Y_{K+l,K+1}^{+} + 1 + \sum_{j=1}^{l}Y_{1,j-1}^{-}h_{j}^{\downarrow}\left(Y_{K+j-1,j+1}^{+}\right)Y_{K+j-1,K+1}^{+} \\
&= 1 + \sum_{j=1}^{l+1}Y_{1,j-1}^{-}h_{j}^{\downarrow}\left(Y_{K+j-1,j+1}^{+}\right)Y_{K+j-1,K+1}^{+},
\end{aligned}$$

and we have proved Lemma 5.3.3. $\qquad\square$

By this preceding lemma with $l = K - 1$, we see

$$Y_{1,K-1}^{-}Y_{2K-1,K+1}^{+} = 1 + \sum_{j=1}^{K-1}Y_{1,j-1}^{-}h_{j}^{\downarrow}\left(Y_{K+j-1,j+1}^{+}\right)Y_{K+j-1,K+1}^{+},$$

which takes care of our base case. Proceeding to the inductive step, we calculate

$$\begin{aligned}
&\left(Y_{1,K-1}^{-}\right)^{m+1}\left(Y_{2K-1,K+1}^{+}\right)^{m+1} \\
&= \left(Y_{1,K-1}^{-}\right)^{m}Y_{1,K-1}^{-}Y_{2K-1,K+1}^{+}\left(Y_{2K-1,K+1}^{+}\right)^{m} \\
&= \left(Y_{1,K-1}^{-}\right)^{m}\left(1 + \sum_{j=1}^{K-1}Y_{1,j-1}^{-}h_{j}^{\downarrow}\left(Y_{K+j-1,j+1}^{+}\right)Y_{K+j-1,K+1}^{+}\right)\left(Y_{2K-1,K+1}^{+}\right)^{m} \\
&= \left(Y_{1,K-1}^{-}\right)^{m}\left(Y_{2K-1,K+1}^{+}\right)^{m} + \left(Y_{1,K-1}^{-}\right)^{m}\left(\sum_{j=1}^{K-1}Y_{1,j-1}^{-}h_{j}^{\downarrow}\left(Y_{K+j-1,j+1}^{+}\right)Y_{K+j-1,K+1}^{+}\right)\left(Y_{2K-1,K+1}^{+}\right)^{m} \\
&= 1 + \sum_{s=0}^{m-1}\left(Y_{1,K-1}^{-}\right)^{s}\left(\sum_{j=1}^{K-1}Y_{1,j-1}^{-}h_{j}^{\downarrow}\left(Y_{K+j-1,j+1}^{+}\right)Y_{K+j-1,K+1}^{+}\right)\left(Y_{2K-1,K+1}^{+}\right)^{s} \\
&\quad + \left(Y_{1,K-1}^{-}\right)^{m}\left(\sum_{j=1}^{K-1}Y_{1,j-1}^{-}h_{j}^{\downarrow}\left(Y_{K+j-1,j+1}^{+}\right)Y_{K+j-1,K+1}^{+}\right)\left(Y_{2K-1,K+1}^{+}\right)^{m} \\
&= 1 + \sum_{s=0}^{m}\left(Y_{1,K-1}^{-}\right)^{s}\left(\sum_{j=1}^{K-1}Y_{1,j-1}^{-}h_{j}^{\downarrow}\left(Y_{K+j-1,j+1}^{+}\right)Y_{K+j-1,K+1}^{+}\right)\left(Y_{2K-1,K+1}^{+}\right)^{s},
\end{aligned}$$

which completes our proof. $\qquad\square$

Lemma 5.3.4. For $0 \leq l \leq K$ and $l + 1 \leq q \leq K + 1$, we have

$$Y_{1,l}^- Y_{K+l,q}^+ = Y_K Y_{K-1,q}^+ \prod_{t=1}^{l} h_t \left(b_{q-1}^{-1} Y_{q-1,t+1}^+ Y_{K+t-1,q}^+ \right) \tag{5.11}$$

Proof. If $l = 0$, then the product in (5.11) is empty, and so we are left with

$$Y_{1,0}^- Y_{K,q}^+ = Y_K Y_{K-1,q}^+$$

$$a_0 Y_{K,q}^+ = b_K^{-1} Y_{K,q}^+$$

which is true by (5.2), and so (5.11) holds. Otherwise, note that

$$Y_{1,l}^- Y_{K+l,q}^+ = Y_{1,l-1}^- Y_l a_l b_{K+l} Y_{K+l} Y_{K+l-1,q}^+$$

$$= Y_{1,l-1}^- Y_l Y_{K+l} Y_{K+l-1,q}^+$$

$$= Y_{1,l-1}^- h_l \left(Y_{K+l-1,l+1}^+ \right) Y_{K+l-1,q}^+$$

$$= Y_{1,l-1}^- \left(\sum_{i=0}^{d_l} \left(Y_{K+l-1,l+1}^+ \right)^i \right) Y_{K+l-1,q}^+$$

$$= Y_{1,l-1}^- \left(\sum_{i=0}^{d_l} \left(Y_{K+l-1,q}^+ b_{q-1}^{-1} Y_{q-1,l+1}^+ \right)^i Y_{K+l-1,q}^+ \right)$$

$$= Y_{1,l-1}^- \left(\sum_{i=0}^{d_l} Y_{K+l-1,q}^+ \left(b_{q-1}^{-1} Y_{q-1,l+1}^+ Y_{K+l-1,q}^+ \right)^i \right)$$

$$= Y_{1,l-1}^- Y_{K+l-1,q}^+ \left(\sum_{i=0}^{d_l} \left(b_{q-1}^{-1} Y_{q-1,l+1}^+ Y_{K+l-1,q}^+ \right)^i \right)$$

$$= Y_{1,l-1}^- Y_{K+l-1,q}^+ h_l \left(b_{q-1}^{-1} Y_{q-1,l+1}^+ Y_{K+l-1,q}^+ \right).$$

Repeating this, we find

$$Y_{1,l}^- Y_{K+l,q}^+ = Y_{1,0}^- Y_{K,q}^+ \prod_{t=1}^{l} h_t \left(b_{q-1}^{-1} Y_{q-1,t+1}^+ Y_{K+t-1,q}^+ \right)$$

$$= a_0 b_K Y_K Y_{K-1,q}^+ \prod_{t=1}^{l} h_t \left(b_{q-1}^{-1} Y_{q-1,t+1}^+ Y_{K+t-1,q}^+ \right)$$

$$= Y_K Y_{K-1,q}^+ \prod_{t=1}^{l} h_t \left(b_{q-1}^{-1} Y_{q-1,t+1}^+ Y_{K+t-1,q}^+ \right).$$

$\square$

Lemma 5.3.5. For $m \geq 0$,

$$\left(Y_{1,K-1}^- \right)^m \left(Y_{2K-1,K+1}^+ \right)^m = 1 + Y_K \sum_{s=0}^{m-1} \sum_{j=1}^{K-1} \left(Y_{K-1,1}^+ \right)^s Y_{K-1,j+1}^+ A \left(j, s \right)$$

where

$$A\left(j,s\right) = \left(\prod_{t=1}^{j-1} h_t\left(b_j^{-1}Y_{j,t+1}^+ Y_{K+t-1,j+1}^+\right)\right) h_j^{\downarrow\downarrow}\left(Y_{K+j-1,j+1}^+\right) Y_{K+j-1,K+1}^+ \left(Y_{2K-1,K+1}^+\right)^s.$$

(5.12)

Proof. From Lemma 5.3.4 with $l = j - 1$ and $q = j + 1$,

$$\begin{aligned}
Y_{1,j-1}^- h_j^\downarrow\left(Y_{K+j-1,j+1}^+\right) &= Y_{1,j-1}^- Y_{K+j-1,j+1}^+ h_j^{\downarrow\downarrow}\left(Y_{K+j-1,j+1}^+\right) \\
&= Y_K Y_{K-1,j+1}^+ \left(\prod_{t=1}^{j-1} h_t\left(b_j^{-1}Y_{j,t+1}^+ Y_{K+t-1,j+1}^+\right)\right) h_j^{\downarrow\downarrow}\left(Y_{K+j-1,j+1}^+\right),
\end{aligned}$$

and so, by Lemma 5.2.2,

$$\begin{aligned}
&\left(Y_{1,K-1}^-\right)^s Y_{1,j-1}^- h_j^\downarrow\left(Y_{K+j-1,j+1}^+\right) \\
&= \left(Y_{1,K-1}^-\right)^s Y_K Y_{K-1,j+1}^+ \left(\prod_{t=1}^{j-1} h_t\left(b_j^{-1}Y_{j,t+1}^+ Y_{K+t-1,j+1}^+\right)\right) h_j^{\downarrow\downarrow}\left(Y_{K+j-1,j+1}^+\right) \\
&= Y_K \left(Y_{K-1,1}^+\right)^s Y_{K-1,j+1}^+ \left(\prod_{t=1}^{j-1} h_t\left(b_j^{-1}Y_{j,t+1}^+ Y_{K+t-1,j+1}^+\right)\right) h_j^{\downarrow\downarrow}\left(Y_{K+j-1,j+1}^+\right).
\end{aligned}$$

Recalling Lemma 5.3.2, regrouping, and substituting the above expression gives

$$\begin{aligned}
&\left(Y_{1,K-1}^-\right)^m \left(Y_{2K-1,K+1}^+\right)^m \\
&= 1 + \sum_{s=0}^{m-1} \left(Y_{1,K-1}^-\right)^s \left(\sum_{j=1}^{K-1} Y_{1,j-1}^- h_j^\downarrow\left(Y_{K+j-1,j+1}^+\right) Y_{K+j-1,K+1}^+\right) \left(Y_{2K-1,K+1}^+\right)^s \\
&= 1 + \sum_{s=0}^{m-1}\sum_{j=1}^{K-1} \left(\left(Y_{1,K-1}^-\right)^s Y_{1,j-1}^- h_j^\downarrow\left(Y_{K+j-1,j+1}^+\right)\right) Y_{K+j-1,K+1}^+ \left(Y_{2K-1,K+1}^+\right)^s \\
&= 1 + \sum_{s=0}^{m-1}\sum_{j=1}^{K-1} \left(Y_K\left(Y_{K-1,1}^+\right)^s Y_{K-1,j+1}^+ \left(\prod_{t=1}^{j-1} h_t\left(b_j^{-1}Y_{j,t+1}^+ Y_{K+t-1,j+1}^+\right)\right) h_j^{\downarrow\downarrow}\left(Y_{K+j-1,j+1}^+\right)\right) \\
&\qquad\qquad\qquad\qquad \cdot Y_{K+j-1,K+1}^+ \left(Y_{2K-1,K+1}^+\right)^s \\
&= 1 + Y_K \sum_{s=0}^{m-1}\sum_{j=1}^{K-1} \left(Y_{K-1,1}^+\right)^s Y_{K-1,j+1}^+ A\left(j,s\right),
\end{aligned}$$

as desired. $\qquad\square$

We now have our final expression for $\left(Y_{1,K-1}^-\right)^m \left(Y_{2K-1,K+1}^+\right)^m$. So, by (??) and

Lemma 5.3.5, we find

$$Y_K^{-1}\left(Y_{2K-1,K+1}^+\right)^{d_K} = Y_0\left(Y_{2K-1,K+1}^+\right)^{d_K} - Y_K^{-1}\sum_{m=1}^{d_0} P_{0,m}\left(Y_{2K-1,K+1}^+\right)^{d_K-m}$$

$$- Y_K^{-1}\sum_{m=1}^{d_0} P_{0,m}\left(Y_K\sum_{s=0}^{m-1}\sum_{j=1}^{K-1}\left(Y_{K-1,1}^+\right)^s Y_{K-1,j+1}^+ A\left(j,d_K+s-m\right)\right)$$

$$= Y_0\left(Y_{2K-1,K+1}^+\right)^{d_K} - Y_K^{-1}\sum_{m=1}^{d_0} P_{0,m}\left(Y_{2K-1,K+1}^+\right)^{d_K-m}$$

$$- \sum_{m=1}^{d_0}\sum_{s=0}^{m-1}\sum_{j=1}^{K-1} P_{0,m}\left(Y_{K-1,1}^+\right)^s Y_{K-1,j+1}^+ A\left(j,d_K+s-m\right).$$

Now, by (5.8),

$$Y_{2K} = Y_K^{-1}\sum_{i=0}^{d_K-1} P_{K,i}\left(Y_{2K-1,K+1}^+\right)^i + Y_0\left(Y_{2K-1,K+1}^+\right)^{d_K} - Y_K^{-1}\sum_{m=1}^{d_0} P_{0,m}\left(Y_{2K-1,K+1}^+\right)^{d_K-m}$$

$$- \sum_{m=1}^{d_0}\sum_{s=0}^{m-1}\sum_{j=1}^{K-1} P_{0,m}\left(Y_{K-1,1}^+\right)^s Y_{K-1,j+1}^+ A\left(j,d_K+s-m\right).$$

Using the facts that $h_0 = h_K$ (so $d_0 = d_K$ and $P_{0,i} = P_{K,i}$ for all i) and h_0 is palindromic, we find

$$\sum_{m=1}^{d_0} P_{0,m}\left(Y_{2K-1,K+1}^+\right)^{d_0-m} = \sum_{m=1}^{d_0} P_{0,d_0-m}\left(Y_{2K-1,K+1}^+\right)^{d_K-m}$$

$$= \sum_{i=0}^{d_0-1} P_{0,i}\left(Y_{2K-1,K+1}^+\right)^i$$

$$= \sum_{i=0}^{d_K-1} P_{K,i}\left(Y_{2K-1,K+1}^+\right)^i.$$

We then get the desired cancellation of terms, and can write

$$Y_{2K} = Y_0\left(Y_{2K-1,K+1}^+\right)^{d_K} - \sum_{m=1}^{d_0}\sum_{s=0}^{m-1}\sum_{j=1}^{K-1} P_{0,m}\left(Y_{K-1,1}^+\right)^s Y_{K-1,j+1}^+ A\left(j,d_K+s-m\right).$$

$$(5.13)$$

We have finally shown that $Y_{2K} \in \mathcal{A}_0$, which proves our main theorem. $\qquad\square$

As a corollary, we see that these recursions have the Laurent phenomenon: each Y_n is a noncommutative Laurent polynomial in $Y_1,\ldots,Y_K$.

5.4 Maple programs

The Maple package `NonComChecker` was written to accompany this paper. It is freely available at `math.rutgers.edu/~russell2/papers/recursions13.html`.

The main function, `VerifyPaper(H,x)`, inputs a H, a list of K polynomials in a variable x, which are taken to be $h_1(x), h_2(x), \ldots, h_K(x) = h_0(x)$. It then simplifies Y_{2K} using a list of equations that it generates, including (5.1), (5.2), (5.3), (5.5), and (5.6), and verifies that the result equals (5.13).

5.5 The Laurent phenomenon in two-dimensional non-commutative recurrences

5.5.1 Introduction

Di Francesco [24] has recently considered the two-dimensional non-commutative recursion defined by

$$T_{j,k+1}T_{j,k-1}^{\bullet} = 1 + T_{j-1,k}T_{j+1,k}^{\bullet} \tag{5.14}$$

together with the relations

$$T_{j,k-1}^{-1}T_{j+1,k} = T_{j+1,k}^{\bullet}\left(T_{j,k-1}^{\bullet}\right)^{-1} \tag{5.15}$$

$$T_{j-1,k}T_{j,k-1}^{-1} = \left(T_{j,k-1}^{\bullet}\right)^{-1}T_{j-1,k}^{\bullet}. \tag{5.16}$$

His work proved the positive Laurent phenomenon for this system: that all T and $T^{\bullet}$ are noncommutative Laurent polynomials in the initial data with positive integral coefficients, and showed that this was an integrable system. In this note, we will attempt to expand this to more general recursions, following the methods first used in [15] and later expanded on in [52]. Our methods will only prove the Laurent phenomenon

To-do:

comment on $\bullet$

Discuss commutative analogues in history

break into props/lems/thms

5.5.2 Preliminaries

Let $h_1(x)$ and $h_2(x)$ be two monic palindromic polynomials. Then, we define

$$T_{j,k+1}T^\bullet_{j,k-1} = h_1\left(T_{j-1,k}T^\bullet_{j+1,k}\right) \quad \text{if } k \text{ is odd} \tag{5.17}$$

$$T_{j,k+1}T^\bullet_{j,k-1} = h_2\left(T_{j-1,k}T^\bullet_{j+1,k}\right) \quad \text{if } k \text{ is even,} \tag{5.18}$$

together with the same relations in (5.15)–(5.16). By applying the anti-automorphism $\bullet$ to the above equations, we also have

$$T_{j,k-1}T^\bullet_{j,k+1} = h_1\left(T_{j+1,k}T^\bullet_{j-1,k}\right) \quad \text{if } k \text{ is odd} \tag{5.19}$$

$$T_{j,k-1}T^\bullet_{j,k+1} = h_2\left(T_{j+1,k}T^\bullet_{j-1,k}\right) \quad \text{if } k \text{ is even.} \tag{5.20}$$

Finally, we also find that, through use of (5.15) and (5.16) (for $a \in \{1,2\}$),

$$T_{j,k+1} = h_a\left(T_{j-1,k}T^\bullet_{j+1,k}\right)\left(T^\bullet_{j,k-1}\right)^{-1} \tag{5.21}$$

$$= \left(T^\bullet_{j,k-1}\right)^{-1} h_a\left(T^\bullet_{j-1,k}T_{j+1,k}\right) \tag{5.22}$$

$$T^\bullet_{j,k-1}T_{j,k+1} = h_a\left(T^\bullet_{j-1,k}T_{j+1,k}\right) \tag{5.23}$$

$$T^\bullet_{j,k+1} = \left(T_{j,k-1}\right)^{-1} h_a\left(T_{j+1,k}T^\bullet_{j-1,k}\right) \tag{5.24}$$

$$= h_a\left(T^\bullet_{j+1,k}T_{j-1,k}\right)\left(T_{j,k-1}\right)^{-1} \tag{5.25}$$

$$T^\bullet_{j,k+1}T_{j,k-1} = h_a\left(T^\bullet_{j+1,k}T_{j-1,k}\right). \tag{5.26}$$

5.5.3 Main results

We assume (without loss of generality) that k is even. We find

$$T_{j,k+1} = h_1\left(T_{j-1,k}T^\bullet_{j+1,k}\right)\left(T^\bullet_{j,k-1}\right)^{-1} \tag{5.27}$$

$$= \sum_{i=0}^{d_1-1}\left(T_{j-1,k}T^\bullet_{j+1,k}\right)^i\left(T^\bullet_{j,k-1}\right)^{-1} + \left(T_{j-1,k}T^\bullet_{j+1,k}\right)^{d_1}\left(T^\bullet_{j,k-1}\right)^{-1}. \tag{5.28}$$

But furthermore,

$$T_{j,k-3}T^\bullet_{j,k-1} = h_1\left(T_{j+1,k-2}T^\bullet_{j-1,k-2}\right) \tag{5.29}$$

$$T_{j,k-3} = \left(1 + h_1^\downarrow\left(T_{j+1,k-2}T^\bullet_{j-1,k-2}\right)\right)\left(T^\bullet_{j,k-1}\right)^{-1} \tag{5.30}$$

$$\left(T^\bullet_{j,k-1}\right)^{-1} = T_{j,k-3} - h_1^\downarrow\left(T_{j+1,k-2}T^\bullet_{j-1,k-2}\right)\left(T^\bullet_{j,k-1}\right)^{-1} \tag{5.31}$$

$$\left(T_{j-1,k}T^\bullet_{j+1,k}\right)^{d_1}\left(T^\bullet_{j,k-1}\right)^{-1} = \left(T_{j-1,k}T^\bullet_{j+1,k}\right)^{d_1}T_{j,k-3} - \left(T_{j-1,k}T^\bullet_{j+1,k}\right)^{d_1}h_1^\downarrow\left(T_{j+1,k-2}T^\bullet_{j-1,k-2}\right)\left(T^\bullet_{j,k-1}\right)^{-1}, \tag{5.32}$$

so, substituting into our previous expression, we find

$$T_{j,k+1} = \sum_{i=0}^{d_1-1} \left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^i \left(T_{j,k-1}^{\bullet}\right)^{-1} + \left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{d_1} T_{j,k-3} \qquad (5.33)$$

$$- \left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{d_1} h_1^{\downarrow} \left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right) \left(T_{j,k-1}^{\bullet}\right)^{-1}. \qquad (5.34)$$

Now,

$$\left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{d_1} h_1^{\downarrow} \left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right) \left(T_{j,k-1}^{\bullet}\right)^{-1} \qquad (5.35)$$

$$= \left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{d_1} \sum_{m=1}^{d_1} \left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^m \left(T_{j,k-1}^{\bullet}\right)^{-1} \qquad (5.36)$$

$$= \sum_{m=1}^{d_1} \left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{d_1-m} \left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^m \left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^m \left(T_{j,k-1}^{\bullet}\right)^{-1}. \qquad (5.37)$$

We would like to calculate $\left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^m \left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^m$. In case $m = 0$, this is just 1. If $m = 1$, then

$$T_{j-1,k}T_{j+1,k}^{\bullet}T_{j+1,k-2}T_{j-1,k-2}^{\bullet} = T_{j-1,k} \left(T_{j+1,k}^{\bullet}T_{j+1,k-2}\right) T_{j-1,k-2}^{\bullet} \qquad (5.38)$$

$$= T_{j-1,k}h_2 \left(T_{j+2,k-1}^{\bullet}T_{j,k-1}\right) T_{j-1,k-2}^{\bullet} \qquad (5.39)$$

$$= T_{j-1,k}T_{j-1,k-2}^{\bullet} + T_{j-1,k}h_2^{\downarrow} \left(T_{j+2,k-1}^{\bullet}T_{j,k-1}\right) T_{j-1,k-2}^{\bullet} \qquad (5.40)$$

$$= h_2 \left(T_{j-2,k-1}T_{j,k-1}^{\bullet}\right) + T_{j-1,k}h_2^{\downarrow} \left(T_{j+2,k-1}^{\bullet}T_{j,k-1}\right) T_{j-1,k-2}^{\bullet} \qquad (5.41)$$

$$= 1 + h_2^{\downarrow} \left(T_{j-2,k-1}T_{j,k-1}^{\bullet}\right) + T_{j-1,k}h_2^{\downarrow} \left(T_{j+2,k-1}^{\bullet}T_{j,k-1}\right) T_{j-1,k-2}^{\bullet} \qquad (5.42)$$

Now,

$$\left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{m+1} \left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^{m+1} \qquad (5.43)$$

$$= \left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^m \left(T_{j-1,k}T_{j+1,k}^{\bullet}T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right) \left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^m \qquad (5.44)$$

$$= \left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^m \left(1 + h_2^{\downarrow} \left(T_{j-2,k-1}T_{j,k-1}^{\bullet}\right) + T_{j-1,k}h_2^{\downarrow} \left(T_{j+2,k-1}^{\bullet}T_{j,k-1}\right) T_{j-1,k-2}^{\bullet}\right) \left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^m \qquad (5.45)$$

$$= \left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^m \left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^m \qquad (5.46)$$

$$+ \left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^m \left(h_2^{\downarrow} \left(T_{j-2,k-1}T_{j,k-1}^{\bullet}\right) + T_{j-1,k}h_2^{\downarrow} \left(T_{j+2,k-1}^{\bullet}T_{j,k-1}\right) T_{j-1,k-2}^{\bullet}\right) \left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^m \qquad (5.47)$$

Thus, by an inductive argument, we conclude that

$$\left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{m}\left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^{m} \tag{5.48}$$

$$= 1 + \sum_{i=0}^{m-1}\left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{i}\left(h_{2}^{\downarrow}\left(T_{j-2,k-1}T_{j,k-1}^{\bullet}\right) + T_{j-1,k}h_{2}^{\downarrow}\left(T_{j+2,k-1}^{\bullet}T_{j,k-1}\right)T_{j-1,k-2}^{\bullet}\right)\left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^{i} \tag{5.49}$$

$$= 1 + \sum_{i=0}^{m-1}\left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{i}\left(T_{j-2,k-1}h_{2}^{\downarrow\downarrow}\left(T_{j,k-1}^{\bullet}T_{j-2,k-1}\right)T_{j,k-1}^{\bullet}\right. \tag{5.50}$$

$$+ T_{j-1,k}T_{j+2,k-1}^{\bullet}h_{2}^{\downarrow\downarrow}\left(T_{j,k-1}T_{j+2,k-1}^{\bullet}\right)T_{j,k-1}T_{j-1,k-2}^{\bullet}\left.\right)\left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^{i} \tag{5.51}$$

$$= 1 + \sum_{i=0}^{m-1}\left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{i}\left(T_{j-2,k-1}h_{2}^{\downarrow\downarrow}\left(T_{j,k-1}^{\bullet}T_{j-2,k-1}\right)T_{j,k-1}^{\bullet}\right. \tag{5.52}$$

$$+ T_{j-1,k}T_{j+2,k-1}^{\bullet}h_{2}^{\downarrow\downarrow}\left(T_{j,k-1}T_{j+2,k-1}^{\bullet}\right)T_{j-1,k-2}T_{j,k-1}^{\bullet}\left.\right)\left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^{i} \tag{5.53}$$

$$= 1 + \sum_{i=0}^{m-1}\left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{i}\left(T_{j-2,k-1}h_{2}^{\downarrow\downarrow}\left(T_{j,k-1}^{\bullet}T_{j-2,k-1}\right)\right. \tag{5.54}$$

$$+ T_{j-1,k}T_{j+2,k-1}^{\bullet}h_{2}^{\downarrow\downarrow}\left(T_{j,k-1}T_{j+2,k-1}^{\bullet}\right)T_{j-1,k-2})T_{j,k-1}^{\bullet}\left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^{i} \tag{5.55}$$

Using (5.16) and (5.15), we calculate

$$T_{j,k-1}^{\bullet}T_{j+1,k-2}T_{j-1,k-2}^{\bullet} = T_{j+1,k-2}^{\bullet}T_{j,k-1}T_{j-1,k-2}^{\bullet} = T_{j+1,k-2}^{\bullet}T_{j-1,k-2}T_{j,k-1}^{\bullet}. \tag{5.56}$$

Hence,

$$T_{j,k-1}^{\bullet}\left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^{i} = \left(T_{j+1,k-2}^{\bullet}T_{j-1,k-2}\right)^{i}T_{j,k-1}^{\bullet} = \left(T_{j+1,k-2}^{\bullet}T_{j-1,k-2}\right)^{i}T_{j,k-1}^{\bullet}, \tag{5.57}$$

and so

$$\left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{m}\left(T_{j+1,k-2}T_{j-1,k-2}^{\bullet}\right)^{m} \tag{5.58}$$

$$= 1 + \sum_{i=0}^{m-1}\left(T_{j-1,k}T_{j+1,k}^{\bullet}\right)^{i}\left(T_{j-2,k-1}h_{2}^{\downarrow\downarrow}\left(T_{j,k-1}^{\bullet}T_{j-2,k-1}\right)\right. \tag{5.59}$$

$$+ T_{j-1,k}T_{j+2,k-1}^{\bullet}h_{2}^{\downarrow\downarrow}\left(T_{j,k-1}T_{j+2,k-1}^{\bullet}\right)T_{j-1,k-2})\left(T_{j+1,k-2}^{\bullet}T_{j-1,k-2}\right)^{i}T_{j,k-1}^{\bullet}. \tag{5.60}$$

Now,

$$\left(T_{j-1,k}T_{j+1,k}^\bullet\right)^{d_1} h_1^\downarrow \left(T_{j+1,k-2}T_{j-1,k-2}^\bullet\right) \left(T_{j,k-1}^\bullet\right)^{-1} \tag{5.61}$$

$$= \sum_{m=1}^{d_1} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^{d_1-m} \left(1 + \sum_{i=0}^{m-1} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^i \left(T_{j-2,k-1}h_2^{\downarrow\downarrow} \left(T_{j,k-1}^\bullet T_{j-2,k-1}\right)\right.\right. \tag{5.62}$$

$$+ T_{j-1,k}T_{j+2,k-1}^\bullet h_2^{\downarrow\downarrow} \left(T_{j,k-1}T_{j+2,k-1}^\bullet\right) T_{j-1,k-2}\right) \left(T_{j+1,k-2}^\bullet T_{j-1,k-2}\right)^i T_{j,k-1}^\bullet\right) \left(T_{j,k-1}^\bullet\right)^{-1} \tag{5.63}$$

$$= \sum_{m=1}^{d_1} \sum_{i=0}^{m-1} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^{d_1-m} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^i \left(T_{j-2,k-1}h_2^{\downarrow\downarrow} \left(T_{j,k-1}^\bullet T_{j-2,k-1}\right)\right. \tag{5.64}$$

$$+ T_{j-1,k}T_{j+2,k-1}^\bullet h_2^{\downarrow\downarrow} \left(T_{j,k-1}T_{j+2,k-1}^\bullet\right) T_{j-1,k-2}\right) \left(T_{j+1,k-2}^\bullet T_{j-1,k-2}\right)^i + \sum_{m=1}^{d_1} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^{d_1-m} \left(T_{j,k-1}^\bullet\right)^{-1}. \tag{5.65}$$

Putting it all together, we see that

$$T_{j,k+1} = \sum_{i=0}^{d_1-1} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^i \left(T_{j,k-1}^\bullet\right)^{-1} + \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^{d_1} T_{j,k-3} \tag{5.66}$$

$$- \sum_{m=1}^{d_1} \sum_{i=0}^{m-1} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^{d_1-m} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^i \left(T_{j-2,k-1}h_2^{\downarrow\downarrow} \left(T_{j,k-1}^\bullet T_{j-2,k-1}\right)\right. \tag{5.67}$$

$$+ T_{j-1,k}T_{j+2,k-1}^\bullet h_2^{\downarrow\downarrow} \left(T_{j,k-1}T_{j+2,k-1}^\bullet\right) T_{j-1,k-2}\right) \left(T_{j+1,k-2}^\bullet T_{j-1,k-2}\right)^i \tag{5.68}$$

$$- \sum_{m=1}^{d_1} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^{d_1-m} \left(T_{j,k-1}^\bullet\right)^{-1}. \tag{5.69}$$

But,

$$\sum_{m=1}^{d_1} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^{d_1-m} \left(T_{j,k-1}^\bullet\right)^{-1} = \sum_{i=0}^{d_1-1} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^i \left(T_{j,k-1}^\bullet\right)^{-1}, \tag{5.70}$$

so it simplifies to

$$T_{j,k+1} = \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^{d_1} T_{j,k-3} \tag{5.71}$$

$$- \sum_{m=1}^{d_1} \sum_{i=0}^{m-1} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^{d_1-m} \left(T_{j-1,k}T_{j+1,k}^\bullet\right)^i \left(T_{j-2,k-1}h_2^{\downarrow\downarrow} \left(T_{j,k-1}^\bullet T_{j-2,k-1}\right)\right. \tag{5.72}$$

$$+ T_{j-1,k}T_{j+2,k-1}^\bullet h_2^{\downarrow\downarrow} \left(T_{j,k-1}T_{j+2,k-1}^\bullet\right) T_{j-1,k-2}\right) \left(T_{j+1,k-2}^\bullet T_{j-1,k-2}\right)^i. \tag{5.73}$$

No negative powers — Laurent phenomenon!

5.5.4   Further work

It would be interesting to try to expand this work to more general two-dimensional recurrences of greater order - for example, —. Alternatively, it would be interesting to consider three- or higher-dimensional recurrences.

Commutative analogues of these recurrences have already been considered in —.

Chapter 6

Experimental mathematics and integer sequences

The Somos sequences, first studied by Michael Somos, are recurrence relations that surprisingly produce only integers. Their integrality turns out to be a special case of the Laurent phenomenon. Since their initial discovery, additional families of sequences with this property have been discovered. We will discuss methods for searching for new sequences with the Laurent phenomenon - with the conjecturing and proving both automated. Careful examination of the computer-generated proofs in individual cases can then lead to human proofs for new infinite families.

6.1   Introduction and background

In his study of elliptic curves, Michael Somos came across the following intriguing sequences:

. The Somos-4 sequence. Define a sequence $a(n)$ by $a(1) = a(2) = a(3) = a(4) = 1$, and, for $n > 4$, $a(n) = \frac{a(n-1)a(n-3)+a(n-2)^2}{a(n-4)}$.

. The Somos-5 sequence. Define a sequence $a(n)$ by $a(1) = a(2) = a(3) = a(4) = a(5) = 1$, and, for $n > 5$, $a(n) = \frac{a(n-1)a(n-4)+a(n-2)a(n-3)}{a(n-5)}$.

Clearly, both Somos sequences will generate rational numbers. Surprisingly, they both appeared to generate only integers: The Somos-4 sequence begins $1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209$ and the Somos-5 sequence begins $1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, 6161, \ldots$ (A006720 and A006721 in the OEIS [49]). The Somos-4 sequence was proven to be integral in []; the proof examined a series of congruences.

Later, while researching the Somos-4 sequence, Dana Scott discovered the sequence that bears his name.

. Dana Scott's sequence. Define a sequence $a\,(n)$ by $a\,(1) = a\,(2) = a\,(3) = a\,(4) = 1$, and, for $n > 4$, $a\,(n) = \frac{a(n-1)a(n-3)+a(n-2)}{a(n-4)}$.

This sequence is defined the same as the Somos-4 sequence, except that the power of the $a\,(n-2)$ is now 1 instead of 2. (This was supposedly due to a typo by Scott.) Once again, an integral sequence is generated: $1, 1, 1, 1, 2, 3, 5, 13, 22, 41, 111, 191, 361, \ldots$ (A048736 in the OEIS [49]). An interesting aspect of Dana Scott's sequence is the fact that it was not purely quadratic - it contained both linear and quadratic terms, unlike the Somos sequences.

Then, Fomin and Zelevinsky [27] made a major breakthrough in understanding sequences of this type. If, instead of using beginning our sequences with $1, \ldots, 1$, we use formal variables $a(1) = a_1, a(2) = a_2, \ldots, a(K) = a_K$, then we say that a sequence has the Laurent phenomenon if each of its terms is a Laurent polynomial in $a_1, \ldots, a_K$ (that is, each term is a rational function of $a_1, \ldots, a_K$ whose denominator is a monomial). They then provided some quite powerful algebraic machinery for proving Laurentness of sequences, which can be used to prove integrality of all of the above sequences (a proof for Somos-4 is a special case of Example 3.3 in [27]). Of course, Laurentness of a sequence immediately implies integrality of the corresponding sequence with all ones at the beginning (as the numerator will always be an integer, and the denominator will always equal 1).

Let F be a polynomial in $x_1, \ldots, x_{K-1}$, and consider the recurrence defined by

$$a(n)a(n-K) = F(a(n-1), \ldots, a(n-K+1)). \qquad (6.1)$$

If we are discussing integrality of the sequence, we use initial conditions $a(1) = a(2) = \cdots = a(K) = 1$. On the other hand, if we are discussing Laurentness, we use formal variables $a(1) = a_1, a(2) = a_2, \ldots, a(K) = a_K$. Laurentness of a sequence implies integrality in the corresponding case, and thus is more general, but in a certain sense, integrality results are more interesting, as we can form lots of explicit sequences and see if they are in the OEIS [49] or not.

Fomin and Zelevinsky [27] give sufficient conditions for the resulting recurrence to

satisfy the Laurent phenomenon. They define

$$Q_m = F\left(x_{m+1}, \ldots, x_{K-1}, 0, x_1, \ldots, x_{m-1}\right) \tag{6.2}$$

In other words, x_i is replaced by $x_{(m+i) \mod K}$, with x_{K-m} being replaced by 0. Then, they define a sequence of polynomials $G_{K-1}, \ldots, G_0$ by $G_{K-1} = F$, and then, each G_{m-1} is obtained from G_m by

$$\widetilde{G}_{m-1} = G_m \mid_{x_m \leftarrow \frac{Q_m}{x_m}} \tag{6.3}$$

$$\widetilde{\widetilde{G}}_{m-1} = \frac{\widetilde{G}_m}{L} \tag{6.4}$$

$$G_{m-1} = \frac{\widetilde{\widetilde{G}}_{m-1}}{Q_m^b}, \tag{6.5}$$

where L is a Laurent monomial that makes $\widetilde{\widetilde{G}}_{m-1}$ be a polynomial not divisible by any x_i or non-invertible scalar, and b is the highest power of Q_m that divides $\widetilde{\widetilde{G}}_{m-1}$. Their theorem 3.1 essentially says that a sequence of the form (6.1) has the Laurent phenomenon if the following three conditions for F hold:

1. F is not divisible by any x_i.

2. Each Q_m is an irreducible element of $\mathbb{Z}\left[x_1^{\pm}, x_{K-1}^{\pm}\right]$.

3. $G_0 = F$.

In this work, we only consider quadratic recurrences: those where F is a polynomial of degree at most 2. Furthermore, we consider only the case where each possible term of F appears with a coefficient of either 0 or 1. The theorems and conjectures that now follow were inspired by a computer search of all possible polynomials F of this form for small values of K.

This paper is accompanied by the Maple package SOMOS. It is available at the website for this paper, `math.rutgers.edu/~russell2/papers/somos`. The package contains procedures that, for given d and K, finds all recurrences of order K in the form (6.1) with initial conditions $a(1) = \cdots = a(K) = 1$, where F is a polynomial of degree at most d, and where the coefficient of each term in F is either 0 or 1. Then, for each sequence, the procedure conjectures whether or not it is integral by calculating

out a number of terms (obviously, if at any point it calculates a non-integral term, the sequence is not integral). If the sequence is conjectured to be integral, then it then automatically attempts to prove integrality using the Fomin-Zelevinsky criteria, and, if successful, can output a proof. (Of course, a failed test does not indicate non-integrality or non-Laurentness.) All of the theorems were first conjectured by looking at the output of the program, and all of the proofs were guided by looking at the computer-generated proofs.

6.2 Infinite families of sequences with the Laurent Phenomenon

Throughout this section, we assume that the sequences begin with formal variables $a(1) = a_1, a(2) = a_2, \ldots, a(K) = a_K$, unless otherwise noted.

Theorem 6.2.1. Consider the recurrence

$$a(n)a(n-K) = \sum_{j=1}^{k-1} a\left(n-j\right)^2. \tag{6.6}$$

This sequence has the Laurent phenomenon.

Proof. The corresponding polynomial is

$$F\left(x_1, \ldots, x_{K-1}\right) = \sum_{j=1}^{K-1} x_j^2. \tag{6.7}$$

From (6.2), we have

$$Q_m = \sum_{j=1, j \neq m}^{K-1} x_j^2. \tag{6.8}$$

We have $G_{K-1} = F$, and, for all i from 1 to $K - 1$, we have

$$\tilde{G}_{K-i-1} = \sum_{j=1, j \neq K-i}^{K-1} x_j^2 + \frac{\left(\sum_{j=1, j \neq K-i}^{K-1} x_j^2\right)^2}{x_{K-i}^2} \tag{6.9}$$

$$\tilde{\tilde{G}}_{K-i-1} = \tilde{G}_{K-i-1} \, x_{K-i}^2 \tag{6.10}$$

$$= \left(\sum_{j=1, j \neq K-i}^{K-1} x_j^2\right)\left(x_{K-i}^2 + \sum_{j=1, j \neq K-i}^{K-1} x_j^2\right) \tag{6.11}$$

$$= Q_{K-i} \sum_{j=1}^{K-1} x_j^2 \tag{6.12}$$

$$G_{K-i-1} = \frac{\tilde{\tilde{G}}_{K-i-1}}{Q_{K-i}} \tag{6.13}$$

$$= \sum_{j=1}^{K-1} x_j^2 = F, \tag{6.14}$$

which is easily verified by induction. Thus, we have $G_0 = F$, so the sequence is integral, by the Fomin-Zelevinsky criteria. $\square$

Remark 6.2.2. The corresponding sequences for $K = 3, 4, 5$ appear in OEIS [49] as A064098, A072878, and A072879, but the sequences for $K \geq 6$ do not appear to be included.

Theorem 6.2.3. Consider the recurrence

$$a(n)a(n - K) = 1 + \sum_{j=1}^{k-1} a(n - j) + a(n-1)a(n - K + 1). \tag{6.15}$$

This sequence has the Laurent phenomenon.

Proof. Our polynomial is

$$F(x_1, \ldots, x_{K-1}) = 1 + \sum_{j=1}^{K-1} x_j + x_1 x_{K-1}. \tag{6.16}$$

Then, from (6.2), we have

$$Q_m = \begin{cases} 1 + \sum_{j=1, j \neq m}^{K-1} x_j \text{ if } m = 1 \text{ or } m = K - 1 \\ 1 + \sum_{j=1, j \neq m}^{K-1} x_j + x_{m+1}x_{m-1} \text{ otherwise.} \end{cases} \tag{6.17}$$

Now, $G_{K-1} = F$, and

Claim 6.2.4. For $2 \leq j \leq K - 1$, we have $G_{K-j} = 1 + x_1 + x_{K-1}$.

Proof. Proceeding by backwards induction,

$$\widetilde{G}_{K-2} = G_{K-1} \big|_{x_{K-1} \leftarrow \frac{Q_m}{x_m}} \tag{6.18}$$

$$= 1 + \sum_{j=1}^{K-2} x_j + (1 + x_1) \frac{1 + \sum_{j=1}^{K-2} x_j}{x_{K-1}} \tag{6.19}$$

$$\widetilde{\widetilde{G}}_{K-2} = x_{K-1} \, \widetilde{G}_{K-1} \tag{6.20}$$

$$= \left(1 + \sum_{j=1}^{K-2} x_j \right) (1 + x_1 + x_{K-1}) \tag{6.21}$$

$$= (1 + x_1 + x_{K-1}) \, Q_{K-1} \tag{6.22}$$

$$G_{K-2} = \frac{\widetilde{\widetilde{G}}}{Q_{K-1}} = 1 + x_1 + x_{K-1}. \tag{6.23}$$

Suppose inductive hypothesis.

$$\widetilde{G}_{K-j-1} = G_{K-j} \big|_{x_{K-j} \leftarrow \frac{Q_{K-j}}{x_{K-j}}} \tag{6.24}$$

$$= 1 + x_1 + x_{K-1} = G_{K-j} \tag{6.25}$$

$$\widetilde{\widetilde{G}}_{K-j-1} = \widetilde{G}_{K-j-1} \tag{6.26}$$

$$G_{K-j-1} = \widetilde{\widetilde{G}}_{K-j-1} = 1 + x_1 + x_{K-1} \tag{6.27}$$

This completes the proof. $\qquad\square$

So, we are left with $G_1 = 1 + x_1 + x_{K-1}$. Then,

$$\widetilde{G}_0 = G_1 \big|_{x_1 \leftarrow \frac{Q_1}{x_1}} \tag{6.28}$$

$$= 1 + x_{K-1} + \frac{1 + \sum_{j=2}^{K-1} x_j}{x_1} \tag{6.29}$$

$$\widetilde{\widetilde{G}}_0 = x_1 \, \widetilde{G}_0 \tag{6.30}$$

$$= x_1 (1 + x_{K-1}) + 1 + \sum_{j=2}^{K-1} x_j \tag{6.31}$$

$$= 1 + \sum_{j=1}^{K-1} x_j x_1 x_{K-1} \tag{6.32}$$

$$G_0 = \widetilde{\widetilde{G}}_0 = 1 + \sum_{j=1}^{K-1} x_j x_1 x_{K-1} = G_{K-1}. \tag{6.33}$$

Thus, we have the Laurent phenomenon. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.2.5. For $K = 3$, we get the sequence $1, 1, 1, 4, 10, 55, 154, 868, 2449, 13825, 39025, \ldots,$ for $K = 4$, we find $1, 1, 1, 1, 5, 13, 33, 217, 617, 1633, 10813, \ldots$, and for $K = 5$, we have $1, 1, 1, 1, 1, 6, 16, 41, 106, 806, 2311, \ldots$. None of these sequences appear to be in the OEIS [49].

Theorem 6.2.6. Let K be odd. Consider the recurrence

$$a(n)a(n - K) = \sum_{j=1}^{\frac{K-1}{2}} a\left(n - 2j + 1\right) a\left(n - 2j\right). \tag{6.34}$$

This sequence has the Laurent phenomenon.

Proof. We find that $G_{K-1} = F = \sum_{j=1}^{(K-1)/2} x_{2j-1} x_{2j}$.

$$Q_m = \begin{cases} \sum_{j=1}^{\frac{m-1}{2}} x_{2j-1} x_{2j} + \sum_{j=(m+1)/2}^{\frac{K-3}{2}} x_{2j} x_{2j+1} & \text{for } m \text{ odd} \\ \sum_{j=1}^{\frac{m}{2}-1} x_{2j} x_{2j+1} + \sum_{j=m/2}^{\frac{K-3}{2}} x_{2j+1} x_{2j+2} & \text{for } m \text{ even.} \end{cases} \tag{6.35}$$

Claim 6.2.7. For l odd, we have

$$G_l = x_l Q_{l+1} + x_{l+1} Q_l. \tag{6.36}$$

Proof. Backwards induction: begin with G_{K-2}. Here, we have $Q_{K-1} = \sum_{j=1}^{(K-3)/2} x_{2j} x_{2j+1}$.

$$\widetilde{G}_{K-2} = G_{K-1} \Big|_{x_{K-1} \leftarrow \frac{Q_{K-1}}{x_{K-1}}} \tag{6.37}$$

$$= \sum_{j=1}^{(K-3)/2} x_{2j-1} x_{2j} + \frac{x_{K-2}}{x_{K-1}} Q_{K-1} \tag{6.38}$$

$$\widetilde{\widetilde{G}}_{K-2} = x_{K-1} \widetilde{G}_{K-2} \tag{6.39}$$

$$= x_{K-1} \sum_{j=1}^{(K-3)/2} x_{2j-1} x_{2j} + x_{K-2} Q_{K-1} \tag{6.40}$$

$$= x_{K-1} Q_{K-2} + x_{K-2} Q_{K-1} \tag{6.41}$$

$$G_{K-2} = \widetilde{\widetilde{G}}_{K-2} = x_{K-2} Q_{K-1} + x_{K-1} Q_{K-2}. \tag{6.42}$$

This takes care of the base case.

$$\widetilde{G}_{l-1} = G_l \mid_{x_l \leftarrow \frac{Q_l}{x_l}} \tag{6.43}$$

$$= \frac{Q_l}{x_l} \left(\sum_{j=1}^{\frac{l-3}{2}} x_{2j} x_{2j+1} + \frac{x_{l-1}}{x_l} Q_l + \sum_{j=(l+1)/2}^{\frac{K-3}{2}} x_{2j+1} x_{2j+2} \right) + x_{l+1} Q_l \tag{6.44}$$

$$\widetilde{\widetilde{G}}_{l-1} = x_l^2 \, \widetilde{G}_{l-1} \tag{6.45}$$

$$= x_{l-1} Q_l^2 + x_l Q_l \left(\sum_{j=1}^{\frac{l-3}{2}} x_{2j} x_{2j+1} + \sum_{j=(l+1)/2}^{\frac{K-3}{2}} x_{2j+1} x_{2j+2} \right) + x_l^2 x_{l+1} Q_l \tag{6.46}$$

$$= x_{l-1} Q_l^2 + x_l Q_l \left(\sum_{j=1}^{\frac{l-3}{2}} x_{2j} x_{2j+1} + x_l x_{l+1} + \sum_{j=(l+1)/2}^{\frac{K-3}{2}} x_{2j+1} x_{2j+2} \right) \tag{6.47}$$

$$= x_{l-1} Q_l^2 + x_l Q_l \left(\sum_{j=1}^{\frac{l-3}{2}} x_{2j} x_{2j+1} + \sum_{j=(l-1)/2}^{\frac{K-3}{2}} x_{2j+1} x_{2j+2} \right) \tag{6.48}$$

$$= Q_l \left(x_{l-1} Q_l + x_l Q_{l-1} \right) \tag{6.49}$$

$$G_{l-1} = \frac{\widetilde{\widetilde{G}}_{l-1}}{Q_l} = x_{l-1} Q_l + x_l Q_{l-1} \tag{6.50}$$

$$\widetilde{G}_{l-2} = G_{l-1} \mid_{x_{l-1} \leftarrow \frac{Q_{l-1}}{x_{l-1}}} \tag{6.51}$$

$$= \frac{Q_{l-1}}{x_{l-1}} \left(\sum_{j=1}^{\frac{l-3}{2}} x_{2j-1} x_{2j} + \frac{x_{l-2}}{x_{l-1}} Q_{l-1} + \sum_{j=(l+1)/2}^{\frac{K-3}{2}} x_{2j} x_{2j+1} \right) + x_l Q_{l-1} \tag{6.52}$$

$$\widetilde{\widetilde{G}}_{l-2} = x_{l-1}^2 \, \widetilde{G}_{l-2} \tag{6.53}$$

$$= x_{l-2} Q_{l-1}^2 + x_{l-1} Q_{l-1} \left(\sum_{j=1}^{\frac{l-3}{2}} x_{2j-1} x_{2j} + \sum_{j=(l+1)/2}^{\frac{K-3}{2}} x_{2j} x_{2j+1} \right) + x_{l-1}^2 x_l Q_{l-1} \tag{6.54}$$

$$= x_{l-2} Q_{l-1}^2 + x_{l-1} Q_{l-1} \left(\sum_{j=1}^{\frac{l-3}{2}} x_{2j-1} x_{2j} + x_{l-1} x_l + \sum_{j=(l+1)/2}^{\frac{K-3}{2}} x_{2j} x_{2j+1} \right) \tag{6.55}$$

$$= x_{l-2} Q_{l-1}^2 + x_{l-1} Q_{l-1} \left(\sum_{j=1}^{\frac{l-3}{2}} x_{2j-1} x_{2j} + \sum_{j=(l-1)/2}^{\frac{K-3}{2}} x_{2j} x_{2j+1} \right) \tag{6.56}$$

$$= Q_{l-1} \left(x_{l-2} Q_{l-1} + x_{l-1} Q_{l-2} \right) \tag{6.57}$$

$$G_{l-2} = \frac{\widetilde{\widetilde{G}}_{l-2}}{Q_{l-1}} \tag{6.58}$$

$$= x_{l-2} Q_{l-1} + x_{l-1} Q_{l-2}. \tag{6.59}$$

□

Thus, we end up with $G_1 = x_1 Q_2 + x_2 Q_1$. We have $Q_1 = \sum_{j=1}^{(K-3)/2} x_{2j} x_{2j+1}$, and can then find

$$\widetilde{G}_0 = G_1 \big|_{x_1 \leftarrow \frac{Q_1}{x_1}} \tag{6.60}$$

$$= \frac{Q_1}{x_1} Q_2 + x_2 Q_1 \tag{6.61}$$

$$\widetilde{\widetilde{G}}_0 = x_1 \, \widetilde{G}_0 \tag{6.62}$$

$$= Q_1 \left(x_1 x_2 + Q_2 \right) \tag{6.63}$$

$$G_0 = \frac{\widetilde{\widetilde{G}}_0}{Q_1} \tag{6.64}$$

$$= x_1 x_2 + \sum_{j=1}^{\frac{(K-3)}{2}} x_{2j+1} x_{2j+2} \tag{6.65}$$

$$= \sum_{j=1}^{\frac{(K-1)}{2}} x_{2j-1} x_{2j} = G_{K-1}. \tag{6.66}$$

□

Remark 6.2.8. For $K = 5$, this is simply the Somos-5 sequence. For $K = 7$, we have $1, 1, 1, 1, 1, 1, 1, 3, 5, 17, 89, 1529, 136169, \ldots$, and for $K = 9$, we get $1, \ldots, 1, 4, 7, 31, 223, 6943, \ldots$. Neither of these are in the OEIS [49].

Theorem 6.2.9. Let K be odd, and consider the recurrence

$$a(n)a(n - K) = 1 + \sum_{i=1}^{K-2} a(n - i) a(n - i - 1). \tag{6.67}$$

This sequence has the Laurent phenomenon.

Proof. The corresponding polynomial is

$$F(x_1, \ldots, x_{K-1}) = 1 + \sum_{i=1}^{K-2} x_i x_{i+1}. \tag{6.68}$$

From (6.2), we have

$$Q_m = 1 + \sum_{i=1}^{m-2} x_i x_{i+1} + \sum_{i=m+1}^{K-2} x_i x_{i+1}. \tag{6.69}$$

The first two criteria easily check, so we are left with checking the third criterion.

Claim 6.2.10. For $j \in \left\{1, \ldots, \frac{K-1}{2}\right\}$, we have $G_{K-2j} = x_{K-2j} + x_{K-2j+1}$.

Proof. We will proceed by induction. We find

$$\tilde{G}_{K-2} = G_{K-1}\big|_{x_{K-1} \leftarrow \frac{Q_{K-1}}{x_{K-1}}} \tag{6.70}$$

$$= 1 + \sum_{i=1}^{K-3} x_i x_{i+1} + \frac{x_{K-2}}{x_{K-1}} \left(1 + \sum_{i=1}^{K-3} x_i x_{i+1}\right) \tag{6.71}$$

$$\tilde{\tilde{G}}_{K-2} = \tilde{G}_{K-2}\, x_{K-1} \tag{6.72}$$

$$= \left(1 + \sum_{i=1}^{K-3} x_i x_{i+1}\right) (x_{K-2} + x_{K-1}) \tag{6.73}$$

$$= Q_{K-1} (x_{K-2} + x_{K-1}) \tag{6.74}$$

$$G_{K-2} = \frac{\tilde{\tilde{G}}_{K-2}}{Q_{K-1}} = x_{K-2} + x_{K-1}. \tag{6.75}$$

This establishes the base case, $j = 1$. Now,

$$\widetilde{G}_{K-2j-1} = G_{K-2j}\big|_{x_{K-2j} \leftarrow \frac{Q_{K-2j}}{x_{K-2j}}} \tag{6.76}$$

$$= \frac{1 + \sum_{i=1}^{K-2j-2} x_i x_{i+1} + \sum_{i=K-2j+1}^{K-2} x_i x_{i+1}}{x_{K-2j}} + x_{K-2j+1} \tag{6.77}$$

$$\widetilde{\widetilde{G}}_{K-2j-1} = \widetilde{G}_{K-2j-1}\, x_{K-2j} \tag{6.78}$$

$$= 1 + \sum_{i=1}^{K-2j-2} x_i x_{i+1} + \sum_{i=K-2j+1}^{K-2} x_i x_{i+1} + x_{K-2j} x_{K-2j+1} \tag{6.79}$$

$$= 1 + \sum_{i=1}^{K-2j-2} x_i x_{i+1} + \sum_{i=K-2j}^{K-2} x_i x_{i+1} \tag{6.80}$$

$$G_{K-2j-1} = \frac{\widetilde{\widetilde{G}}_{K-2j-1}}{Q_{K-2j}^0} = 1 + \sum_{i=1}^{K-2j-2} x_i x_{i+1} + \sum_{i=K-2j}^{K-2} x_i x_{i+1} \tag{6.81}$$

$$\widetilde{G}_{K-2j-2} = G_{K-2j-1}\big|_{x_{K-2j-1} \leftarrow \frac{Q_{K-2j-1}}{x_{K-2j-1}}} \tag{6.82}$$

$$= 1 + \sum_{i=1}^{K-2j-3} x_i x_{i+1} + \frac{x_{K-2j-2}}{x_{K-2j-1}} \left(1 + \sum_{i=1}^{K-2j-3} x_i x_{i+1} + \sum_{i=K-2j}^{K-2} x_i x_{i+1} \right) + \sum_{i=K-2j}^{K-2} x_i x_{i+1} \tag{6.83}$$

$$\widetilde{\widetilde{G}}_{K-2j-2} = \widetilde{G}_{K-2j-2}\, x_{K-2j-1} \tag{6.84}$$

$$= \left(1 + \sum_{i=1}^{K-2j-3} x_i x_{i+1} + \sum_{i=K-2j}^{K-2} x_i x_{i+1} \right) (x_{K-2j-2} + x_{K-2j-1}) \tag{6.85}$$

$$= Q_{K-2j-1} (x_{K-2j-2} + x_{K-2j-1}) \tag{6.86}$$

$$G_{K-2j-2} = \frac{\widetilde{\widetilde{G}}_{K-2}}{Q_{K-2j-1}} = x_{K-2j-2} + x_{K-2j-1}. \tag{6.87}$$

This completes the inductive step. $\qquad\qquad\square$

Consequently, in the case $j = \frac{K-1}{2}$, we find $G_{K-2\frac{K-1}{2}} = G_1 = x_1 + x_2$. Then,

$$\widetilde{G}_0 = G_1\big|_{x_1 \leftarrow \frac{Q_1}{x_1}} \tag{6.88}$$

$$= \frac{1 + \sum_{i=2}^{K-2} x_i x_{i+1}}{x_1} + x_2 \tag{6.89}$$

$$\widetilde{\widetilde{G}}_0 = \widetilde{G}_0\, x_1 \tag{6.90}$$

$$= 1 + \sum_{i=2}^{K-2} x_i x_{i+1} + x_1 x_2 \tag{6.91}$$

$$= 1 + \sum_{i=1}^{K-2} x_i x_{i+1} \tag{6.92}$$

$$G_0 = \frac{\widetilde{\widetilde{G}}_0}{Q_1^0} = 1 + \sum_{i=1}^{K-2} x_i x_{i+1} = F. \tag{6.93}$$

Thus, we conclude that this sequence has the Laurent phenomenon. $\square$

Remark 6.2.11. The sequence for $K = 3$, $1, 1, 1, 2, 3, 7, 11, 26, 41, 97, 153, 362, 571, \ldots$,

is A005246 in the OEIS [49]. None of the sequences for $K = 5$: $1, 1, 1, 1, 1, 4, 7, 34, 271, 9481, 644701, \ldots$,

for $K = 7$: $1, 1, 1, 1, 1, 1, 1, 6, 11, 76, 911, 70146, 63973151, \ldots$, and, finally, for $K = 9$:

$1, 1, 1, 1, 1, 1, 1, 1, 1, 8, 15, 134, 2143, 289304, 620267775, \ldots$, are in the OEIS.

Theorem 6.2.12. Let K be even and i be such that $0 < i < \frac{K}{2}$. Consider the recurrence

$$a(n)a(n-K) = a\left(n - \frac{K}{2}\right) + a\left(n - i\right)a\left(n - K + i\right). \tag{6.94}$$

This sequence has the Laurent phenomenon.

Proof. The corresponding polynomial is

$$F(x_1, \ldots, x_{K-1}) = x_{K/2} + x_i x_{K-i}. \tag{6.95}$$

From (6.2), we have

$$Q_m = \begin{cases} x_{m+K/2} + x_{m+i}x_{m+K-i}, & m < i \\ x_{K/2+i}, & m = i \\ x_{m+K/2} + x_{m+i}x_{m-i}, & i < m < \frac{K}{2} \\ x_{K/2-i}x_{K/2+i}, & m = \frac{K}{2} \\ x_{m-K/2} + x_{m+i}x_{m-i}, & \frac{K}{2} < m < K - i \\ x_{K/2-i}, & m = K - i \\ x_{m-K/2} + x_{m+i-K}x_{m-i}, & K - i < m. \end{cases} \tag{6.96}$$

The first two criteria easily check, so we are left with checking the third criterion.

Claim 6.2.13. For $1 \leq j \leq i$, we have $G_{K-j} = x_{K/2} + x_i x_{K-i}$.

Proof. We will proceed by induction. By definition, $G_{K-1} = F = x_{K/2} + x_i x_{K-i}$, which shows that our base case is satisfied. Assuming the inductive hypothesis, we have

$$\widetilde{G}_{K-1-j} = G_{K-j}\big|_{x_{K-j} \leftarrow \frac{Q_{K-j}}{x_{K-j}}} \tag{6.97}$$

$$= x_{K/2} + x_i x_{K-i} \tag{6.98}$$

$$\widetilde{\widetilde{G}}_{K-1-j} = \widetilde{G}_{K-1-j} \tag{6.99}$$

$$G_{K-1-j} = \widetilde{\widetilde{G}}_{K-1-j} =_{K/2} + x_i x_{K-i}. \tag{6.100}$$

This proves the inductive step, and hence the claim. $\qquad\square$

Claim 6.2.14. For $i < j \leq \frac{K}{2}$, we have $G_{K-j} = x_{K/2-i}x_i + x_{K/2}x_{K-i}$.

Proof. From the previous claim, we have $G_{K-i} = x_{K/2} + x_i x_{K-i}$. Thus,

$$\widetilde{G}_{K-i-1} = G_{K-i}\big|_{x_{K-i} \leftarrow \frac{Q_{K-i}}{x_{K-i}}} \tag{6.101}$$

$$= x_{K/2} + \frac{x_i x_{K/2-i}}{x_{K-i}} \tag{6.102}$$

$$\widetilde{\widetilde{G}}_{K-i-1} = x_{K-i}\, \widetilde{G}_{K-j1} \tag{6.103}$$

$$= x_{K/2}x_{K-i} + x_i x_{K/2-i} \tag{6.104}$$

$$G_{K-i-1} = \widetilde{\widetilde{G}}_{K-i-1} = x_{K/2}x_{K-i} + x_i x_{K/2-i}. \tag{6.105}$$

This takes care of the base case. Then, assuming the inductive hypothesis, we have

$$\tilde{G}_{K-1-j} = G_{K-j}\big|_{x_{K-j} \leftarrow \frac{Q_{K-j}}{x_{K-j}}} \tag{6.106}$$

$$= x_{K/2} x_{K-i} + x_i x_{K/2-i} \tag{6.107}$$

$$\tilde{\tilde{G}}_{K-1-j} = \tilde{G}_{K-i-1} \tag{6.108}$$

$$G_{K-1-j} = \tilde{\tilde{G}}_{K-1-j} = x_{K/2} x_{K-i} + x_i x_{K/2-i}, \tag{6.109}$$

which finishes the proof. $\square$

Claim 6.2.15. For $\frac{K}{2} < j \leq K - i$, we have $G_{K-j} = x_{K/2} x_i + x_{K-i} x_{K/2+i}$.

Proof. From our previous claim, we have $G_{K/2} = x_{K/2} x_{K-i} + x_i x_{K/2-i}$. Then,

$$\tilde{G}_{K/2-1} = G_{K/2}\big|_{x_{K/2} \leftarrow \frac{Q_{K/2}}{x_{K/2}}} \tag{6.110}$$

$$= x_i x_{K/2-i} + \frac{x_{K-i} x_{K/2-i} x_{K/2+i}}{x_{K/2}} \tag{6.111}$$

$$\tilde{\tilde{G}}_{K/2-1} = \frac{x_{K/2}}{x_{K/2-i}} \tilde{G}_{K/2-1} \tag{6.112}$$

$$= x_i x_{K/2} + x_{K-i} x_{K/2+i} \tag{6.113}$$

$$G_{K/2-1} = \tilde{\tilde{G}}_{K/2-1} = x_i x_{K/2} + x_{K-i} x_{K/2+i}, \tag{6.114}$$

which takes care of the base case. For the inductive step:

$$\tilde{G}_{K-1-j} = G_{K-j}\big|_{x_{K-j} \leftarrow \frac{Q_{K-j}}{x_{K-j}}} \tag{6.115}$$

$$= x_i x_{K/2} + x_{K-i} x_{K/2+i} \tag{6.116}$$

$$\tilde{\tilde{G}}_{K-1-j} = \tilde{G}_{K-j-1} = x_i x_{K/2} + x_{K-i} x_{K/2+i} \tag{6.117}$$

$$G_{K-1-j} = \tilde{\tilde{G}}_{K-j-1} = x_i x_{K/2} + x_{K-i} x_{K/2+i}, \tag{6.118}$$

which completes the proof of the claim. $\square$

Claim 6.2.16. For $K - i < j \leq K - 1$, we have $G_{K-j} = x_{K/2} + x_i x_{K-i}$.

Proof. Base case: from before, $G_i = G_{K-(K-i)} = x_{K/2} x_i + x_{K-i} x_{K/2+i}$. Thus,

$$\widetilde{G}_{i-1} = G_i \mid_{x_i \leftarrow \frac{Q_i}{x_i}} \tag{6.119}$$

$$= x_{K/2} \frac{x_{K/2+i}}{x_i} + x_{K-i} x_{K/2+i} \tag{6.120}$$

$$\widetilde{\widetilde{G}}_{i-1} = \frac{x_i}{x_{K/2+i}} \widetilde{G}_{i-1} \tag{6.121}$$

$$= x_{K/2} + x_{K-i} x_i \tag{6.122}$$

$$G_{i-1} = \widetilde{\widetilde{G}}_{i-1} = x_{K/2} + x_{K-i} x_i. \tag{6.123}$$

This finishes the base case. We move on to the inductive step:

$$\widetilde{G}_{K-j-1} = G_{K-j} \mid_{x_{K-j} \leftarrow \frac{Q_{K-j}}{x_{K-j}}} \tag{6.124}$$

$$= x_{K/2} + x_{K-i} x_i \tag{6.125}$$

$$\widetilde{\widetilde{G}}_{K-j-1} = \widetilde{G}_{K-j-1} = x_{K/2} + x_{K-i} x_i \tag{6.126}$$

$$G_{K-j-1} = \widetilde{\widetilde{G}}_{K-j-1} = x_{K/2} + x_{K-i} x_i. \tag{6.127}$$

This completes the proof of the claim.

$\square$

Thus, by using $j = K-1$ in the last claim, we find $G_0 = x_{K/2} + x_{K-i} x_i = G_{K-1}$. $\quad\square$

Theorem 6.2.17. Consider the recurrence

$$a(n)a(n-K) = 1 + a(n-j)a(n-K+j) \tag{6.128}$$

for $1 \le j \le \frac{K}{2}$. This sequence has the Laurent phenomenon.

Proof. The corresponding polynomial is

$$F(x_1, \ldots, x_{K-1}) = 1 + x_j x_{K-j}. \tag{6.129}$$

From (6.2), we have

$$Q_m = \begin{cases} 1 \text{ if } m = j \text{ or } m = K - j \\ \\ 1 + x_{m+j(mod K)} x_{K+m-j(mod K)} \text{ otherwise.} \end{cases} \tag{6.130}$$

The first two conditions trivially check. (Forgive the messiness that follows, I promise all the work is correct, but I haven't had time to polish the exposition.) For the third, we have $G_{K-1} = 1 + x_j x_{K-j}$, and then $G_{K-i-1} = G_{K-1}$ for $i < j$ (because x_i does not appear, see (6.2)). Then, for the j case, we have

$$\widetilde{G}_{K-j-1} = 1 + \frac{x_j}{x_{K-j}} \tag{6.131}$$

$$\widetilde{\widetilde{G}}_{K-j-1} = x_{K-j} + x_j \tag{6.132}$$

$$G_{K-j-1} = x_{K-j} + x_j. \tag{6.133}$$

After that, we will have $G_{K-i-1} = G_{K-j-1}$ for $j < i < K - j$, and in the $K - j$ case, we have

$$\widetilde{G}_{j-1} = x_{K-j} + \frac{1}{x_j} \tag{6.134}$$

$$\widetilde{\widetilde{G}}_{j-1} = x_{K-j} x_j + 1 \tag{6.135}$$

$$G_{j-1} = 1 + x_j x_{K-j}. \tag{6.136}$$

Finally, we will have $G_{K-i-1} = G_{j-1}$ for $i > K - j$, so $G_0 = 1 + x_j x_{K-j} = F$. $\qquad\square$

Remark 6.2.18. This construction seems to give lots of fairly simple sequences, just about all of which do not appear in OEIS [49]. For example, the case where $K = 4$, $F = 1 + x_1 x_3$ produces the sequence $1, 1, 1, 1, 2, 3, 4, 9, 14, 19, 43$, which is not in the OEIS. However, these sequences are probably C-finite, and thus not as interesting.

6.3 General thoughts, conjectures, and directions for further work

We now present a couple of more general conjectures about the structure of polynomials F that give

Conjecture 6.3.1. (Symmetry Conjecture) If F is a polynomial of degree at most two and every coefficient zero or one that satisfies the three conditions of Fomin and Zelevinsky's Theorem 3.1, then we have $F(x_1, x_2, \ldots, x_{K-2}, x_{K-1}) = F(x_{K-1}, x_{K-2}, \ldots, x_2, x_1)$.

Conjecture 6.3.2. (Squared terms in odd recurrences conjecture) If F is a polynomial of degree at most two and every coefficient zero or one that satisfies the three conditions

of Fomin and Zelevinsky's Theorem 3.1, and K is odd (so that F has an even number of variables), then either every possible squared term $(x_1^2, x_2^2, \ldots, x_{K-1}^2)$ appears in F, or none of them do.

It does appear that Fomin and Zelevinsky's Theorem 3.1 is not a necessary condition:

Conjecture 6.3.3. The sequence with $K = 4, F = (1 + x_1)(1 + x_3)$ generates an

This sequence begins

$$1, 1, 1, 1, 4, 10, 22, 115, \ldots \tag{6.137}$$

It is not in the OEIS, but its integrality has been verified for the first 5000 terms. It fails the second Fomin-Zelevinsky criterion.

There are also many other cases where the sequences appear to be integral (after calculating out the first 60 terms or so, at which point it becomes computationally intensive), and the F polynomial satisfies the first two Fomin-Zelevinsky criteria, but fails the third. For example, the sequence with $K = 4, F = 1 + x_1 x_2 + x_2 x_3$ appears to be integral (and, in fact, appears in OEIS [49] - but without a reference to a proof of integrality.

In her thesis [34], Emilie Hogan used three different methods to prove that nonlinear recurrences gave integral solutions: a direct argument using elementary number theory, demonstratring that the recurrence in question actually satisfied a linear recurrence, and Fomin and Zelevinsky's Laurent phenomenon techniques [27]. Other papers use techniques from the theory of elliptic curves.

It is unclear what methods could be used to show that sequences actually fail to be integral, other than simply calculating out terms until a non-integral term is found. However, this could be difficult, as there is no known bound for how far you would have to go (at least, I don't know one). For example, the recurrence of the form (6.1) with $K = 4$ and $F = x_1 x_3 + x_2 (x_1 + x_2 + x_3)$ is integral through the first 27 terms, but the 28th term fails to be integral (and is roughly 10^{1200}). On the other hand, its Laurentness does fail at the ninth term - which is actually the first place that it could fail. I do not know if this always happens - if Laurentness always fails immediately. Furthermore, we

know that Laurentness implies integrality. I wonder if the two conditions can be shown to be equivalent.

Fomin and Zelevinsky also did much work looking at two- and three-dimensional recurrences.

6.4 Acknowledgments

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