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WZ Certification for Abel-Type Identities and Askey's Positivity Conjecture

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by

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ABSTRACT

WZ Certification for Abel-Type Identities and Askey's Positivity Conjecture

by John E. Majewicz

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Advisor: Doron Zeilberger

To prove that $\sum_k F(n,k) = r(n)$, one can find a recurrence relation for which both $A(n) := \sum_k F(n,k)$ and $r(n)$ are a solution. If J is the order of the recurrence, $J+1$ initial conditions also need to be checked. This is the idea of WZ theory which has streamlined the proofs of a large class of identities, namely those in which the summand $F(n,k)$ is hypergeometric.

We show that WZ theory is extendable to Abel-type summation identities which includes the hypergeometric summation identities. The classical example, from which the name is derived, is Abel's identity:

$$\sum \binom{n}{k} r(r+k)^{k-1}(s+n-k)^{n-k} = (r+s+n)^n.$$

More specifically, we show that every Abel-type sum, (the left-hand side of Abel's

identity, for example), satisfies a non-trivial linear recurrence relation, regardless of whether or not it has a closed form.

The other part of the paper is concerned with a positivity question. We conjecture positivity of the McLaurin coefficients of the rational function $\phi(r,s,t,u) := 1/R(r,s,t,u)$, where

$$R := (1-r)[(1-s)+(1-t)+(1-u)] + (1-s)[(1-t)+(1-u)] + (1-t)(1-u).$$

Although the truth of this conjecture has not been established, strong evidence that suggests that it is true is given. In addition, recent positivity results, related to similar rational functions, are summarized.

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DEDICATION

This dissertation is dedicated to my parents, Frederick and Linda Majewicz, from whom I learned the meaning of unconditional love.

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CHAPTER 1
INTRODUCTION

In this paper, two different problems are examined. The first one deals with the question of extending the so-called WZ theory from hypergeometric summation identities to the more inclusive realm of Abel-type identities. The first section of this chapter is concerned with the ground breaking work completed by Herbert Wilf and Doron Zeilberger in the field of algorithmic proof theory. Major results are presented, and they provide a framework for the results we obtained which are presented in Chapter 2.

Suppose one is interested in whether or not certain rational functions give rise to power series with positive, or non-negative, power series coefficients. This is what we will mean when we say "positivity problems". The second section of this chapter is devoted to some past results related to positivity problems. In the third chapter, we present some results we have been able to obtain related to a re-formulated version of a conjecture of Richard Askey and George Gasper.

1.1 WZ Theory

WZ theory is without a doubt a major contribution to the subject of algorithmic proof theory. The beautiful procedures developed by Wilf and Zeilberger [WZ92], [WZ96] have made it possible to prove, in a uniform manner, thousands of summation identities

$$\sum_k F(n,k) = r(n). \tag{1.1}$$

Some of the identities can be rather complicated, but the WZ theory reduces the proofs to routinely verifiable rational function identities. The theory relies on the fact that the sums $A(n) := \sum_k F(n,k)$ satisfy linear recurrence relations with coefficients that are polynomials in n. Originally, it was thought that for the WZ theory to work, $F(n,k)$ had to be hypergeometric. One of the goals of this paper is to show that this is no longer true.

Definition 1.1 *A function is hypergeometric in two summation variables if both* $F(n,k+1)/F(n,k)$ *and* $F(n+1,k)/F(n,k)$ *are rational functions of* n *and* k.

The basis of WZ theory involves finding a linear recurrence relation for which both $A(n)$ and $r(n)$ are a solution. After the initial conditions are checked, the truth of (1.1) follows. Discovery of recurrence relations for which $A(n)$ is a solution when $F(n,k)$ is hypergeometric can be accomplished in three different, but related ways: Gosper's algorithm [WZ96], [G78], Sister Celine's technique [F47],[WZ92] and Zeilberger's algorithm [WZ96].

Gosper's algorithm leads to first-order, linear recurrence relations with coefficients identically 1. The ideas of Gosper's algorithm generalize nicely, so the details are discussed in the next section. Both Sister Celine's technique and Zeilberger's algorithm lead to recurrence relations for which the coefficients are non-constant and the recurrences are least first order. We used Sister Celine's technique to extend the results of WZ theory, so it will be discussed in detail in Section 1.1.2.

1.1.1 Recurrences via Gosper's algorithm

The simplest recurrences, to solve are constant-coefficient, first-order recurrences, among which the easiest to solve are those of the form

$$f(n+1) - f(n) = 0. \tag{1.2}$$

If the initial condition $f(0) = $ constant holds, then it follows that $f(n) = $ constant for all $n \geq 1$. As long as $r(n)$ is not identically 0, (1.1) is transformed into this setting, once both sides of (1.1) are divided by $r(n)$ and the summand is redefined accordingly. Of course, if $r(n) = 0$, then there's no need to redefine the summand $F(n,k)$; (1.2) and $f(0) = 0$ lead to the desired conclusion.

The WZ method leads to recurrences of the form (1.2) by means of the so-called proof certificate $R(n,k)$, which is found using Gosper's algorithm [G78], [WZ96]. This rational function has the wonderful property that

$$F(n+1,k) - F(n,k) = G(n,k) - G(n,k+1) \tag{1.3}$$

holds where $G(n,k) := R(n,k)F(n,k)$. If for each integer $n \geq 0$, $lim_{k\to\infty}G(n,k) = 0$, then summing (1.3) over all k leads to (1.2), due to the telescoping on the right-hand side of (1.3).

Definition 1.2 *The pair of functions (F,G) that satisfy (1.3) is referred to as a WZ-pair and the rational function $R(n,k)$ is called the proof certificate.*

Thus, in many instances, existence of the certificate $R(n,k)$ suffices as a proof of an identity. Start by redefining $F(n,k)$, if necessary. Once presented with $R(n,k)$, simply define $G(n,k)$, check that (1.3) holds and see that $A(0) = 1$. This so-called WZ method is illustrated in the following two examples.

Theorem 1.1 $\sum_k \binom{n}{k} = 2^n$.

Proof: $R(n,k) := \frac{k}{(2(k-n-1)}$ □

Theorem 1.2 *(Dixon's Identity)* $\sum_k (-1)^k \binom{n+b}{n+k}\binom{n+c}{c+k}\binom{b+c}{b+k} = \binom{n+b+c}{n,b,c}$

Proof: $R(n,k) := \frac{(k+b)(k+c)}{2(k-n-1)(n+b+c+1)}$ □

These are only two of many examples of the so-called WZ method. To appreciate the real power of the WZ method, and WZ theory in general, refer to [WZ96] for a comprehensive list. Literally, hundreds of proofs of hypergeometric summation identities have been reduced to rational function identities. The proofs are easy in the sense that they can be easily implemented on a computer algebra software package such as Maple.

Of course, non-existence of a WZ-pair does not mean that $\sum_k F(n,k) = r(n)$ is not true. On the contrary, the proof-certificate $R(n,k)$ always exists, but the recurrence on the left-hand side of (1.3) might not be first-order, and it might not have constant coefficients. This is where Sister Celine's technique comes in handy.

1.1.2 k-free recurrences for proper-hypergeometric terms

Sister Celine's technique and Zeilberger's algorithm share a common goal. Namely, the end result is to find $J+1$ polynomials $a_j(n)$, that depend only on n, and a function $G(n,k)$ such that for all n,

$$\sum_{j=0}^{J} a_j(n) F(n-j,k) = G(n,k) - G(n,k+1). \tag{1.4}$$

We haved used Sister Celine's technique to extend the WZ theory from the hypergeometric realm to the more inclusive Abel-type realm. Therefore, the ideas of Sister Celine are presented in detail; the details of Zeilberger's can be found in [WZ96]. In [WZ92], Sister Celine's technique was applied to the class of proper-hypergeometric terms.

Definition 1.3 *A proper-hypergeometric term F_{ph} is a function of the form*

$$F_{ph}(n,k) = P(n,k)\frac{\prod_{r=1}^{pp}(a_r n + b_r k + c_r)!}{\prod_{s=1}^{qq}(u_s n + v_s k + w_s)!}\xi^k,$$

where P is a polynomial and ξ is a parameter. The a's, b's, u's and v's are assumed to be specific integers that do not depend on any other parameters. As in [WZ92], we will say that F_{ph} is well-defined at (n,k) if none of the numbers $\{a_r n + b_r k + c_r\}_1^{pp}$ is a negative integer. We will say that $F_{ph}(n,k) = 0$ if F_{ph} is well-defined at (n,k) and at least one of the numbers $\{u_s n + v_s k + w_s\}_1^{qq}$ is a negative integer, or $P(n,k) = 0$.

Throughout the remainder of this paper, it can be assumed that $F(n,k)$ is a proper-hypergeometric term.

Definition 1.4 *A general function $F(n,k)$ is said to satisfy a non-trivial k-free recurrence relation if there are integers I, J and polynomials $\alpha_{i,j} = \alpha_{i,j}(n)$ that do not depend on k and are not all zero, such that the relation*

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\alpha_{i,j}(n)F(n-j,k-i) = 0$$

holds for all (n,k) in $\mathbb{Z}^2$.

Sister Celine's technique is used to find k-free recurrences for $F(n,k)$. Eventually, recurrences for which $\sum F(n,k)$ is a solution are derived from k-free recurrences. Wilf and Zeilberger borrowed the ideas of Sister Celine and made it precise. Following her lead, they formed the linear combination

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\alpha_{i,j}(n)\frac{F(n-j,k-i)}{F(n,k)}$$

and utilized the fact that the ratios appearing in the summands are well-defined rational functions of n and k. Then a common denominator for the summands can be found and $\alpha_{i,j}(n)$ can be determined so that every coefficient of the numerator polynomial with respect to k vanishes identically. Once the number of unknowns exceeds the number of equations, the possibility of a solution increases. In fact, an upper bound for I and J has been established [WZ92].

Theorem 1.3 *Every proper-hypergeometric term $F(n,k)$ satisfies a nontrivial k-free recurrence relation. There exists such a recurrence with $(I,J) = (I^*,J^*)$ where*

$$J^* := \textstyle\sum_x |b_s| + \sum_s |v_s| \text{ and } I^* := 1 + deg(P) + J^*(\sum_s |a_s| + \sum_s |u_s| - 1).$$

Proof: See [WZ92].

With this information, we are ready to show how this theorem leads to certification of summation identities.

1.1.3 Certification of proper-hypergeometric identities

Existence of a non-trivial, k-free recurrence relation for $F(n,k)$ guarantees that $A(n)$ satisfies a non-trivial, recurrence relation. The recurrence relation is derived from the k-free one. In this section, we show how this is possible.

Theorem 1.4 *Let $F(n,k)$ be a proper-hypergeometric term , and let $(n,k) \in \mathbb{Z}^2$ be a point at which $F(n,k) \neq 0$ and such that $F(n-j,k-i)$ is well-defined for all $0 \leq i \leq I$ and $0 \leq j \leq J$. Then there are polynomials $a_0(n),\ldots,a_J(n)$, not all zero, and a function $G(n,k)$ such that $G(n,k) = R(n,k)F(n,k)$ for some rational function $R(n,k)$ and such that (1.4) holds.*

Proof: The unconventional notation used there is followed here, so N and K denote the backwards shift operators with respect to n and k. Thus, $NG(n,k) = G(n-1,k)$ and $KG(n,k) = G(n,k-1)$.

First, let $H(N,K,n) := \sum_{i,j} \alpha_{i,j}(n)N^jK^i$ denote the k-free recurrence operator annihilating $F(n,k)$. Then $V(N,K,n)$ is defined so that

$$(K-1)V(N,K,n) := H(N,K,n) - H(N,1,n). \tag{1.5}$$

Since $H(N,K,n)$ is a k-free recurrence operator, (1.5) implies

$$H(N,1,n)F(n,k) = (K-1)(-V(N,K,n)F(n,k)). \tag{1.6}$$

Then

$$G(n,k+1) := -V(N,K,n)F(n,k). \tag{1.7}$$

Being a linear combination of terms of the form $\alpha_{i,j}(n)N^jK^tF(n,k)$, for $0 \leq t \leq i$, it follows that G is a rational multiple of F.

Since $H(N,1,n) = \sum_{i,j} \alpha_{i,j}(n)N^j$, it follows that $a_j(n) := \sum_i \alpha_{ij}(n)$ for $j = 0,\ldots,J$; i.e.,

$$H(N,1,n) = \sum_{j=0}^{J} a_j(n)N^j. \tag{1.8}$$

Substituting (1.7) and (1.8) into (1.6) yields (1.4). □

Example 1.1.1 *Let* $F(n,k) := \binom{n}{k}^2$. *Since* F *satisfies the* k*-free recurrence relation*

$$\begin{aligned} nF(n,k) &- (2n-1)F(n-1,k) \\ &- (2n-1)F(n-1,k-1) + (n-1)F(n-2,k) \\ &- 2(n-1)F(n-2,k-1) + (n-1)F(n-2,k-2) = 0, \end{aligned}$$

we must define

$$H(N,K,n) := n-(2n-1)N-(2n-1)NK+(n-1)N^2-2(n-1)N^2K+(n-1)N^2K^2.$$

$H(N,K,n)$ *can be written in the form*

$$n - 2(2n-1)N + (K-1)\{-(2n-1)N + (n-1)N^2(K-1)\},$$

so let $V(N,K,n) := -(2n-1)N + (n-1)N^2(K-1)$. *It follows that*

$$G(n,k+1) := V(N,K,n)F(n,k) = \frac{(2+2k-3n)(n-k)^2}{n^2}F(n,k),$$

and

$$nF(n,k) - 2(2n-1)F(n-1,k) = G(n,k+1) - G(n,k).$$

The process of proving proper-hypergeometric, summation identities via certification procedures follows through similarly. (1.4) is summed over all k yielding the recurrence

$$a_0(n)A(n) + a_1(n)A(n-1) + \ldots + a_J(n)A(n-J) = 0, \qquad (1.9)$$

with $A(n) := \sum_k F(n,k)$. Then the conjectured closed form of $\sum_k F(n,k)$ is shown to satisfy the same recurrence relation with the same $J+1$ initial conditions. The following is actually a continuation of the previous example.

Theorem 1.5 $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$

Proof: Let $H(N,n) := n - 2(2n-1)N$, $R(n,k) := (2+2k-3n)(n-k)^2/n^2$, $G(n,k) := R(n,k)F(n,k)$ and $A(n) := \sum F(n,k)$. Then

$$H(N,n)F(n,k) = G(n,k+1) - G(n,k),$$

which, by summing over all k implies

$$nA(n) - 2(2n-1)A(n-1) = 0$$

for all $n \geq 1$. But $A(0) = 1 = \binom{2\cdot 0}{0}$ and

$$n\binom{2n}{n} - 2(2n-1)\binom{2(n-1)}{n-1} = 0;$$

i.e., $A(n)$ and $\binom{2n}{n}$ satisfy the same recurrence relation with the same initial condition. Therefore, we can conclude that $A(n) = \binom{2n}{n}$ and the theorem follows. □

As we mentioned earlier, WZ theory was thought to be relevant only for proving identities involving hypergeometric summands. In the second chapter, we show that this is no longer the case. This was accomplished by applying Sister Celine's technique to the so-called Abel-type terms $F(n,k,r,s)$ [M96]. We show that this celebrated technique can be used to show that every Abel-type term satisfies a non-trivial, k-free recurrence relation and that the certification procedure can be extended similarly. Before we do that, we introduce a little background of our second topic of this paper.

1.2 Positivity Problems for Rational Functions

The third chapter of this dissertation is devoted to examination of a conjecture of Richard Askey and George Gasper as communicated to them by Hans Lewy [AG72]. Their conjecture concerns the positivity of $A(m_1, m_2, m_3, m_4)$ which are the coefficients of the generating function

$$\frac{1}{(4 - r - s - t - u) \cdot R(r, s, t, u)} \tag{1.10}$$

with

$$R := (1-r)[(1-s)+(1-t)+(1-u)]+(1-s)[(1-t)+(1-u)]+(1-t)(1-u). \tag{1.11}$$

Askey and Gasper claimed that the polynomial $4-r-s-t-u$ in the denominator was needed for positivity.

In [A75], a similar question was posed. This time, the function in question was similar to (1.10), but the factor $4 - r - s - t - u$ is replaced with $(1 - r)(1 - s)(1 - t)(1 - u)$. Again, it was claimed that without this additional factor in the denominator, positivity is not guaranteed.

In the third chapter, a related conjecture is presented. It seems that

$$\phi(r, s, t, u) := \frac{1}{R}$$

does have positive power series coefficients and that additional factors in the denominator are not needed. Evidence related to this conjecture is presented, and the hope is that new interest in the subject area will be sparked. First, a discussion of related problems and results is in order.

1.2.1 Other Positivity Questions

In their study of the numerical stability of a finite difference approximation to the wave equation in two-space, Friedrichs and Lewy needed positivity of the power series [A75] coefficients of

$$\frac{1}{(1-r)((1-s)+(1-t))+(1-s)(1-t)} = \sum_{m_1,m_2,m_3=0}^{\infty} A(m_1,m_2,m_3) r^{m_1} s^{m_2} t^{m_3}.$$

The first person to establish the truth of this statement was Gabor Szegö [S33]. Szegö's proofs utilized special functions.

More specifically, Askey [A75] writes that Szegö showed that

$$A(m_1,m_2,m_3) = \int_0^\infty L_{m_1}(x) L_{m_2}(x) L_{m_3}(x) e^{-3x} dx \tag{1.12}$$

where $L_{m_i}(x)$ are the Laguerre polynomials $L_m^\alpha(x)$ for the case $\alpha = 0$. These polynomials have as a generating function

$$(1-r)^{-\alpha-1}\exp(-xr/(1-r)) = \sum_{m=0}^{\infty} L_m^\alpha(x) r^m$$

and satisfy the orthogonality relation

$$\int_0^\infty L_m^\alpha(x) L_n^\alpha(x) x^\alpha e^{-x} dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{m,n}.$$

Once the problem was transformed into this setting, positivity of the integral in (1.12) was what was needed to establish the truth of the conjecture of Friedrichs and Lewy.

Szegöo is credited with generalizing the problem into one of establishing positivity of the integral

$$\int_0^\infty L_{m_1}^\alpha(x) L_{m_2}^\alpha(x) L_{m_3}^\alpha(x) x^\alpha e^{-3x} dx,$$

which he accomplished for $\alpha \geq -\frac{1}{2}$. This motivated other mathematicians to discover similar integral nonnegativity results and to establish connections with coefficients of other rational functions.

For example, the much stronger result

$$\int_0^\infty x^\alpha e^{-\lambda x} L_{m_1}^\alpha(x) L_{m_2}^\alpha(x) L_{m_3}^\alpha(x) dx \geq 0 \tag{1.13}$$

for $\alpha > -1$ and $\lambda \geq 2$, was shown to be true in the 1972 paper of Richard Askey and George Gasper [AG72]. When $\lambda = 2$, the left-hand side of (1.13) is equal to the coefficient of the generating function

$$\frac{1}{[1-r-s-t+4rst]^{\alpha+1}}. \tag{1.14}$$

A beautiful proof that implies the truth of (1.13) under the assumption $\alpha = 0$, appears in the wonderful 1979 paper of J. Gillis and J. Kleeman [GK79]. Their ingenious proof uses special properties of the generating function (1.14).

Inspired by the same great idea, J. Gillis, B. Reznick and D. Zeilberger used it successfully [GRZ83] to prove positivity of the coefficients of (1.14) for all $\alpha \geq (-5+\sqrt{17})/2$. Details, generalizations and relevance to our problem are presented in the third chapter in the hopes of motivating further exploration of this clever idea of Gillis and Kleeman.

Yet another generalization was achieved in 1978. In their paper [AIK78], Askey, Mourad Ismail and Tom Koornwinder proved that

$$b(m_1,\ldots,m_k;\lambda_1,\ldots,\lambda_k) := \int_0^\infty \prod_{i=1}^k L_{m_i}^\alpha(\lambda_i x)x^\alpha e^{-x}dx \geq 0 \tag{1.15}$$

holds under certain circumstances (Theorem 1). They also showed that the $b(m_1,\ldots,m_k;\lambda_1,\ldots,\lambda_k)$ are equal to the coefficients of the generating function

$$D_k(\alpha) := \frac{\Gamma(\alpha+1)}{\left[\prod_{i=1}^k(1-r_i)+\sum_{i=1}^k \lambda_i r_i \prod_{j\neq i}(1-r_j)\right]^{\alpha+1}}; \tag{1.16}$$

i.e.,

$$D_k(\alpha) = \sum b(m_1,\ldots,m_k;\lambda_1,\ldots,\lambda_k)r_1^{m_1}\ldots r_k^{m_k}.$$

Unfortunately, this setting seems inappropriate for our situation, for if $k=4$, then we need $\sum_{i=1}^4 \lambda_i = 0$ and $\lambda_{i_1}+\lambda_{i_2}+\lambda_{i_3} = 0$ for all subsets $\{i_1,i_2,i_3\} \subset \{1,2,3,4\}$. But this leads to $\lambda_i = 0$ for $i = 1,2,3,4$.

It does not appear that an equivalent integral expression for the coefficients of ϕ has been discovered. In his 1975 monograph [A75], Askey alludes to such an attempt, but expresses his frustration (and this is an understatement) at having been unable to do find a reduction. Nonetheless, it's not inconceivable that such a reduction exists.

1.2.2 Combinatorial Interpretations

Another of the accomplishments to be found in [GK79] is a combinatorial proof of (1.15) for the case $k = 4$, $\lambda_1 = \ldots = \lambda_4 \geq 2$ and $\alpha = 0$. The integral is interpreted combinatorially and there is not much reference to the associated generating function. Perhaps it's possible to achieve a similar interpretation of the coefficients of ϕ even in the absence of an integral formulation.

A somewhat different interpretation of the coefficients of (1.14) was completed by Ismail and Tamhankar [IT79]. The authors applied McMahon's Master Theorem for the case $\alpha = 0$. Transformation of the Askey, Gasper and Lewy conjecture into a combinatorial setting is achieved in the same way, but it seems just as difficult to prove the new one. For completeness' sake, we present our own equivalent conjecture since it's slightly different from the one given in [IT79].

Let

$$C := \begin{pmatrix} \frac{1}{2} & 1 & \sqrt{-12} & -12 \\ \frac{1}{12} & \frac{1}{2} & 1 & -\sqrt{-12} \\ -\frac{\sqrt{-12}}{144} & \frac{1}{12} & \frac{1}{2} & 1 \\ -\frac{1}{144} & \frac{\sqrt{-12}}{144} & \frac{1}{12} & \frac{1}{2} \end{pmatrix}.$$

Then $\det(I - CX) = \frac{1}{6}R$, where

$$X := \begin{pmatrix} r & r & r & r \\ s & s & s & s \\ t & t & t & t \\ u & u & u & u \end{pmatrix}.$$

McMahon's Master Theorem imples that $P(m_1, m_2, m_3, m_4) \geq 0$ if, and only if, the coefficient of $r^{m_1}s^{m_2}t^{m_3}u^{m_4}$ in the expansion of

$$\prod_{i=1}^{4} (c_{i1}r + c_{i2}s + c_{i3}t + c_{i4}u)^{m_i}$$

is non-negative. In the third chapter, results towards establishing the truth of the conjecture are presented.

CHAPTER 2
WZ CERTIFICATION FOR ABEL-TYPE IDENTITIES

In this chapter, we show how Sister Celine's famous technique has once again been used, this time to extend WZ theory to Abel-type summation identities. Generally speaking, the summands of Abel-type identities contain things of the form n^n or k^k. Thus, without something "extra", the tools of WZ theory can not be used directly. However, with very little effort, it can be shown that every Abel-type term also satisfies a non-trivial, k-free recurrence relation and that the certification procedure developed for hypergeometric summation identities extends similarly.

2.1 k-free recurrences for Abel-type terms

We begin with some definitions that also appear in [M96].

Definition 2.1 *An Abel-type term F is a function of the form*

$$F(n,k,r,s) := F_{ph}(n,k)\ (k+r)^{k-1}(n-k+s)^{n-k}r,$$

where F_{ph} is a proper-hypergeometric term.

When $F_{ph}(n,k) := \binom{n}{k}$, then $F(n,k,r,s)$ is the summand of the celebrated Abel identity [A26], [C74], [GKP89]

$$\sum_k \binom{n}{k} r(r+k)^{k-1}(n+s-k)^{n-k} = (n+r+s)^n.$$

The definition of a k-free recurrence relation for a proper-hypergeometric term was made precise in section 1.1.2. We need a similar notion for Abel-type terms. The key to successful implementation of WZ theory lies in allowing for shifts in the parameters r and s.

Definition 2.2 *An Abel-type term $F(n,k,r,s)$ is said to satisfy a nontrivial k-free recurrence relation if there are integers I, J, I_r, I_s and polynomials $\alpha_{i,j,i_r,i_s} = \alpha_{i,j,i_r,i_s}(n,r,s)$ that do not depend on k and are not all zero, such that the relation*

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{i_r=0}^{I_r}\sum_{i_s=0}^{I_s} \alpha_{i,j,i_r,i_s}(n,r,s)F(n-j,k-i,r-i_r,s-i_s) = 0 \qquad (2.1)$$

holds for all (n,k,r,s) in $\mathbb{Z}^4$, whenever F is well-defined at all the arguments that occur.

In general, $F(n-j,k-i,r-i_r,s-i_s)/F(n,k,r,s)$ are not, rational functions of n and k. However, by making r and s "active" parameters, the shifts can be chosen so that Sister Celine's technique is applicable to the resulting ratios; i.e., the shifts are chosen so that the ratios are rational functions of n and k.

More specifically, since F_{ph} is proper-hypergeometric, the ratios

$$\frac{F(n-j,k-i,r+i,s-i+j)}{F(n,k,r,s)} = \frac{F_{ph}(n-j,k-i)}{F_{ph}(n,k)} \frac{(r+i)(n-k+s)^{i-j}}{r(k+r)^i} \qquad (2.2)$$

are well-defined, rational functions of n and k. Thus, Sister Celine's technique can be applied to the sum

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\alpha_{i,j}\frac{F(n-j,k-i,r+i,s+i-j)}{F(n,k,r,s)}$$

$$= \sum_{i=0}^{I}\sum_{j=0}^{J}\alpha_{i,j}\frac{F_{ph}(n-j,k-i)}{F_{ph}(n,k)}\frac{(r+i)(n-k+s)^{i-j}}{r(k+r)^{i}} \qquad (2.3)$$

Then a common denominator for the summands of (2.3) is determined, as well as $\alpha_{i,j}(n,r,s)$ such that every coefficient of the numerator polynomial with respect to k vanishes identically. As before, when the number of unknown coefficients $\alpha_{i,j}(n,r,s)$ exceeds the number of conditions, the possibility of a solution increases, and as before, an upper bound for I and J has been found. The following theorem can be found in [M96], but it is reproduced here for completeness' sake.

Theorem 2.1 *Every Abel-type term F satisfies a nontrivial k-free recurrence relation. Moreover, there exists a recurrence with $(I,J)=(I^*,J^*)$ where*

$$J^* = \sum_r |b_r| + \sum_s |v_s| + 2, \quad I^* = \deg(P) + J^*\Big(\sum_r |a_r| + \sum_s |u_s|\Big).$$

Proof: Start by making the following definitions [WZ92]: for any real number x, $x^+ := \max\{0,x\}$. Then

$$\begin{aligned}
\mathrm{rf}(x,y) &:= \prod_{j=1}^{x}(y+j)\\
\mathrm{prf}(x,y,u) &:= \prod_{j=x+1}^{y}(u+j)\\
\mathrm{ff}(x,u) &:= \prod_{j=0}^{x-1}(u-j)\\
\mathrm{pff}(x,y,u) &:= \prod_{j=x}^{y-1}(u-j)\\
p_1(i,j,k) &:= \prod_{r=1}^{pp}\{\mathrm{rf}((-ja_r-ib_r)^+, a_rn+b_rk+c_r)\\
&\qquad \mathrm{pff}((ja_r+ib_r)^+, J(a_r)^+ + I(b_r)^+, a_rn+b_rk+c_r)\}\\
p_2(i,j,k) &:= \prod_{s=1}^{qq}\{\mathrm{ff}((ju_s+iv_s)^+, u_sn+v_sk+w_s)\\
&\qquad \mathrm{prf}((-ju_s-iv_s)^+, J(-u_s)^+ + I(-v_s)^+, u_sn+v_sk+w_s)\}.
\end{aligned}$$

The next task is to find a common denominator for the summands of (2.3). Collecting all the terms over the denominator, we find that the numerator polynomial is

$$\sum_{i=0}^{I}\sum_{j=1}^{J}\alpha_{i,j}(n,r,s)C(n,k,r,s) + \alpha_{00}(n,r,s)P(n,k)p_1(0,0,k)p_2(0,0,k)r(n-k+s)^J(k+r)^I\xi^I, \quad (2.4)$$

where

$$C(n,k,r,s) = P(n-j,k-i)p_1(i,j,k)p_2(i,j,k)(r+i)(n-k+s)^{J+i-j}(k+r)^{I-i}\xi^{I-i}.$$

The degree of (2.4) as a polynomial in k is at most

$$I\ \Big\{\sum_r |b_r| + \sum_s |v_s|\Big\} + J\Big\{\sum_r |a_r| + \sum_s |u_s|\Big\} + \deg(P) + I + J$$
$$= I\ \Big\{\sum_r |b_r| + \sum_s |v_s| + 1\Big\} + J\Big\{\sum_r |a_r| + \sum_s |u_s| + 1\Big\} + \deg(P). \quad (2.5)$$

Now, the number of $\alpha_{i,j}$'s available is $1+(I+1)J$ and the number of linear, homogeneous equations they must satisfy is (2.5)+1. Hence, a 'nontrivial solution of this system exists if $(I+1)J >$ (2.5). Let $\gamma := \sum_r |b_r| + \sum_s |v_s| + 1$, $\delta := \sum_r |a_r| + \sum_s |u_s| + 1$ and $\epsilon := \deg(P)$. Then we need $(I+1)J > I\gamma + J\delta + \epsilon$. If $J = 1+\gamma$ then this inequality holds provided $I > (\delta-1)(\gamma+1)+\epsilon$. Thus, there exists a nontrivial recurrence of orders

$$J^* = \sum_r |b_r| + \sum_s |v_s| + 2, \quad I^* = \deg(P) + J^*\Big(\sum_r |a_r| + \sum_s |u_s|\Big).$$

This completes the proof of the theorem. □

Now that we know that k-free recurrence relations for Abel-type terms exist, we can use them to extend the WZ theory. In the following section, we prove that it is possible to find a non-trivial, recurrence relation for $A(n,r,s) := \sum_k F(n,k,r,s)$ from a k-free recurrence relation.

2.2 Recurrences relations for Abel-type sums

Recall that, for proper-hypergeometric terms $F_{ph}(n,k)$, $J > 0$ polynomials $a_j(n)$, are sought such that (1.9) holds with $A(n) := \sum_k F(n,k)$. Using a k-free recurrence relation, a function $G(n,k)$ is found such that (1.4) holds, and then due to the telescoping on the right-hand side, summing over all k yields (1.9). We now show that a similar process can be performed on Abel-type terms.

Definition 2.3 *The sum*

$$A(n,r,s) := \sum_{k=0}^{n} F(n,k,r,s) = \sum_{k=0}^{n} F(n,k)(k+r)^{k-1}(n-k+s)^{n-k}r. \tag{2.6}$$

will be referred to as an Abel-type sum.

Theorem 2.2 *Let F be an Abel-type term and A be its associated Abel-type sum. Suppose $(n,k,r,s) \in \mathbb{Z}^4$ is a point at which $F(n,k,r,s) \neq 0$ and such that $F(n-j,k-i,r+i,s+j-i)$ is well-defined for all $0 \leq i \leq I$ and $0 \leq j \leq J$. Then there are Laurent polynomials $a_0(R,S,n,r,s)$, ..., $a_J(R,S,n,r,s)$, not all zero, and a function $G(n,k,r,s)$ such that*

$$\sum_{j=0}^{J} a_j(R,S,n,r,s)F(n-j,k,r,s) = G(n,k,r,s) - G(n,k+1,r,s) \tag{2.7}$$

Hence, by summing over k and using the telescoping on the right-hand side of (2.7), one has

$$\sum_{j=0}^{J} a_j(R,S,n,r,s)A(n-j,r,s) \equiv 0. \tag{2.8}$$

Proof: First,

$$H(N,K,R,S,n,r,s) := \sum_{j=0}^{J}\sum_{i=0}^{I} \alpha_{i,j}(n,r,s)N^jK^iR^{-i}S^{i-j}$$

is defined to be a non-trivial, k-free recurrence operator. Then an operator V is chosen so that

$$(K-1)V(N,K,R,S,n,r,s) := H(N,K,R,S,n,r,s) - H(N,1,R,S,n,r,s). \tag{2.9}$$

Since $K^i - 1 = (K-1)(K^{i-1} + K^{i-2} + \ldots + 1)$, (2.9) implies that

$$V(N,K,R,S,n,r,s) = \sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{t=0}^{i-1} \alpha_{i,j}(n,r,s)N^jK^tR^{-i}S^{i-j}. \tag{2.10}$$

(2.9), together with the fact that H is a k-free recurrence operator which annihilates F, implies that

$$H(N,1,R,S,n,r,s)F(n,k,r,s) = (K-1)(-V(N,K,R,S,n,r,s)F(n,k,r,s)). \tag{2.11}$$

From (2.10), we know that the summands of VF are of the form

$$\begin{aligned} &\alpha_{i,j}(n,r,s)N^jK^tR^{-i}S^{i-j}F(n,k,r,s) \\ = \;&\alpha_{i,j}(n,r,s)R^{-i+t}S^{i-t}\left(N^jK^tR^{-t}S^{-j+t}F(n,k,r,s)\right). \end{aligned} \tag{2.12}$$

It follows, from (2.2), that each $N^jK^tR^{-t}S^{-j+t}F(n,k,r,s)$ is a rational multiple of $F(n,k,r,s)$, say,

$$N^jK^tR^{-t}S^{-j+t}F(n,k,r,s) := P_{j,t}(n,k,r,s)F(n,k,r,s). \tag{2.13}$$

Substitution of (2.13) into (2.12) yields that the summands of VF are of the form

$$Q_{i,j,t}(n,k,r,s) := \alpha_{i,j}(n,r,s)P_{j,t}(n,k,r+i-t,s-i+t)F(n,k,r+i-t,s-i+t).$$

which are not necessarily rational multiples of F. Nonetheless, we define

$$G(n,k+1,r,s) := -\sum_{i=1}^{I}\sum_{j=1}^{J} \alpha_{i,j}(n,r,s)\left[\sum_{t=0}^{i-1} Q_{i,j,t}(n,k,r,s)\right], \tag{2.14}$$

since we only need the telescoping on the right-hand side of (2.7), regardless of whether or not G is a rational multiple of F.

Finally, we note that

$$H(N,1,R,S,n,r,s) = \sum_{j=0}^{J}\sum_{i=0}^{I} \alpha_{i,j}(n,r,s)N^j R^{-i} S^{i-j},$$

leads to the definition

$$a_j(R,S,n,r,s) := \sum_{i=0}^{I} \alpha_{i,j}(n,r,s) R^{-i} S^{i-j}$$

for $j = 0, \ldots, J$. In other words,

$$H(N,1,R,S,n,r,s) = \sum_{j=0}^{J} a_j(R,S,n,r,s)N^j. \tag{2.15}$$

Then (2.7) follows by substituting (2.14) and (2.15) into (2.11). □

2.3 Implementing the WZ Certification

We've been able to find linear recurrences for at least two different Abel-type sums. The first one we consider is a sum which has a known closed form; i.e., Abel's identity. We also present some of the generalizations we were able to discover. The second sum has no known closed form, but the discovered recurrence allows the user an alternative, and possibly faster, way to compute higher order terms.

2.3.1 Abel's identity and generalizations

In the recent past, Abel's identity was known to come in two versions [A26],[GKP89]:

$$A_{10}(n,r,s) := \sum_{k=0}^{n} \binom{n}{k} (r+k)^k (n+s-k)^{n-k} \frac{r}{r+k} = (n+r+s)^n \tag{2.16}$$

and the so-called symmetrized version

$$A_{11}(n,r,s) := \sum_{k=0}^{n} \binom{n}{k} (r+k)^k (n+s-k)^{n-k} \frac{r}{r+k} \cdot \frac{s}{n+s-k} = (n+r+s)^{n-1}(r+s). \tag{2.17}$$

Both can be proved using the techniques developed for this dissertation [M96], [EM96]. The same techniques suggest that these two versions are specific cases of a more general phenomenon.

More specifically, by setting

$$F_{\mu\nu}(n,k,r,s) := \binom{n}{k} (r+k)^{k-\mu} (n+s-k)^{n-k-\nu}$$

and

$$G_{\mu\nu}(n,k,r,s) := s\binom{n-1}{k-1} (r+k)^{k-\mu} (n+s-k)^{n-k-\nu-1},$$

one can show that

$$\begin{aligned} F_{\mu\nu}(n,k,r,s) &- (n+s)F_{\mu\nu}(n-1,k,r,s+1) \\ &- (n+r)F_{\mu\nu}(n-1,k,r+1,s) \\ &+ (n-1)(n+r+s)F_{\mu\nu}(n-2,k,r+1,s+1) \\ &= G_{\mu\nu}(n,k,r,s) - G_{\mu\nu}(n,k+1,r,s) \end{aligned} \tag{2.18}$$

for all μ and ν. By summing both sides of (2.18) over all k, we have that $a_{\mu\nu}(n,r,s) := \sum_{k=0}^{n} F_{\mu\nu}(n,k,r,s)$ satisfies

$$\begin{aligned} a_{\mu\nu}(n,r,s) &- (n+s)a_{\mu\nu}(n-1,r,s+1) \\ &- (n+r)a_{\mu\nu}(n-1,r+1,s) \\ &+ (n-1)(n+r+s)a_{\mu\nu}(n-2,r+1,s+1) = 0 \end{aligned} \tag{2.19}$$

for all $\mu, \nu \geq 1$. Since

$$a_{10}(n,r,s) := A_{10}(n,k,r,s)/r \text{ and } a_{11}(n,r,s) := A_{11}(n,k,r,s)/rs,$$

the truth of (2.16) and (2.17) is established once its shown that both RHS $(2.16)/r$ and RHS $(2.17)/rs$ satisfy (2.19) and the appropriate initial conditions. Here RHS:=right-hand side. Thus, it seems that Abel's identity comes in "infinitely many versions"; i.e., there is one general version which includes the known cases.

Empirical evidence suggests the following closed form for $A_{\mu\nu}(n,k,r,s)$. ([M96] contains a handful of the specific cases.) A conjecture about the general situation is reproduced here. Let $\xi := \mu + \nu - 1$,

$$Q_{\mu\nu}(r,s) := \prod_{l=0}^{\mu}(r+l)^{\mu-l} \cdot \prod_{l=0}^{\nu}(s+l)^{\nu-l},$$

$$P_{\mu\nu}(n,r,s) := \sum_{j=0}^{\mu-1} \frac{n!}{(n-j)!} p_j(r,s),$$

$$b_{\mu\nu}(n,r,s) := \frac{(n+r+s)^{n-\xi} P_{\mu\nu}(n,r,s)}{Q_{\mu\nu}(r,s)},$$

$$A_{\mu\nu}(n,r,s) := a_{\mu\nu}(n,r,s) \cdot Q_{\mu\nu}(r,s)$$

and define the polynomials $p_k(r,s)$ so that, for $0 \le n \le \mu-1$, we have $b_{\mu\nu}(n,r,s) = a_{\mu\nu}(n,r,s)$.

For example, if $n = 0$, then

$$a_{\mu\nu}(0,r,s) = b_{\mu\nu}(0,r,s) = \frac{P_{\mu\nu}(0,r,s)}{Q_{\mu\nu}(r,s)} = \frac{p_0(r,s)}{Q_{\mu\nu}(r,s)}$$

implies $p_0(r,s) = a_{\mu\nu}(0,r,s)Q_{\mu\nu}(r,s) = A_{\mu\nu}(0,r,s)$, and if $n = 1$, then

$$a_{\mu\nu}(1,r,s) = b_{\mu\nu}(1,r,s) = \frac{(1+r+s)^{1-\xi} P_{\mu\nu}(1,r,s)}{Q_{\mu\nu}(r,s)}$$

$$= \frac{(1+r+s)^{1-\xi}[p_0(r,s) + p_1(r,s)]}{Q_{\mu\nu}(r,s)} = \frac{(1+r+s)^{1-\xi}[A_{\mu\nu}(0,r,s) + p_1(r,s)]}{Q_{\mu\nu}(r,s)},$$

implies $p_1(r,s) = (1+r+s)^{\xi-1}A_{\mu\nu}(1,r,s) - A_{\mu\nu}(0,r,s)$, etc. Symmetry permits the assumption $\mu \geq \nu \geq 1$.

Then it seems that $a_{\mu\nu}(n,r,s) = b_{\mu\nu}(n,r,s)$ for $n \geq \mu$; i.e.,

$$A_{\mu\nu}(n,r,s) = (n+r+s)^{n-\xi}P_{\mu\nu}(n,r,s).$$

If $b_{\mu\nu}(n,r,s)$ satisfies (2.19), then the claim is certainly true, since $b_{\mu\nu}(n,r,s)$ was defined to meet the necessary initial conditions. In fact, $b_{\mu\nu}(n,r,s)$ only needs to satisfy two initial conditions, since (2.19) is second-order. No proof that $b_{\mu\nu}(n,r,s)$ satisfies (2.19) for general μ and ν has been discovered.

2.3.2 Computing sums via their recurrences

The next result is presented in the spirit of the fantastic book "A=B" [WZ96].

Theorem 2.3 *If $F_{ph}(n,k) = \binom{n}{k}\binom{b}{k-1}$, $F(n,k,r,s)$ is the related Abel-type term, $A(n,r,s) := \sum_k F(n,k,r,s)$ and*

$$\begin{aligned} H(N,R,S,n,r,s) &:= \\ & n(r+1)(s+1) - (r+1)(n+s)(n+(2n-1)s)NS^{-1} \\ & +r(n+r)(n-b-1)(s+1)NR^{-1} + (n-1)(r+1)(s+n)^2sN^2S^{-2} \\ & -r(n-1)[(2n-2-b+r)s^2 + (n-b-1)(n+r)(2s+1)]N^2S^{-1}R^{-1} \\ & +rs(n-1)(n-2)[s^2+(2n-b-1+r)s+(n-b-1)(n+r)]N^3S^{-2}R^{-1}, \end{aligned}$$

then

$$H(N,R,S,n,r,s)A(n,r,s) = 0. \tag{2.20}$$

Proof:

$$H(N,R,S,n,r,s)F(n,k,r,s) = G(n,k,r+1,s-1) - G(n,k+1,r+1,s-1)$$

where

$$G := brs^2(s+1)\binom{n-1}{k-1}\binom{b-1}{k-2}(k+r)^{k-1}(n-k+s)^{n-k-2}.$$

Summing over all k gives (2.20). □

Thus, $A(n,r,s)$ satisfies a third-order recurrence relation with the initial conditions $A(1,r,s)=1$, $A(2,r,s)=2+b$ and $A(3,r,s)=b^2+2b+3$. This fact can be used as an alternative way to compute $A(n,r,s)$ for $n \geq 4$.

The program `celine` was used to find this recurrence relation, as well as the one used to prove Abel's identity. The details of the program `celine` can be found in Appendix A.

CHAPTER 3

A POSITIVITY CONJECTURE

Recall that $\phi(r,s,t,u) := R^{-1}$ with R given by

$$\begin{aligned} R := (1-r)(1-s) &+ (1-r)(1-t) + (1-r)(1-u) \\ &+ (1-s)(1-t) + (1-s)(1-u) + (1-t)(1-u). \end{aligned}$$

We suspect very strongly that all of the power series coefficients of ϕ are positive. In this chapter we present results obtained towards establishing this fact.

3.1 Empirical evidence and concrete results

Let

$$\phi(r,s,t,u) := \frac{1}{R} = \sum P(m_1,m_2,m_3,m_4) r^{m_1} s^{m_2} t^{m_3} u^{m_4}. \tag{3.1}$$

If one fixes a value of $m_1 = M_1$, $m_2 = M_2$ and $m_3 = M_3$ on the right-hand side of (3.1), but still sums over all m_4, the result is a rational function of u whose coefficients are $P(M_1, M_2, M_3, m_4)$.

For example, $m_1 = 0$, $m_2 = 0$ and $m_3 = 0$ gives

$$\sum_{m_4} P(0,0,0,m_4) u^{m_4} = \frac{1}{1-u},$$

whereas $m_1 = 0$, $m_2 = 0$ and $m_3 = 1$ yields the function

$$\sum_{m_4} P(0,0,1,m_4) u^{m_4} = \frac{1}{3}\left(\frac{1}{(1-u)^2} + \frac{2}{1-u}\right),$$

The coefficients of both functions are all positive. In other words, ϕ has infinitely many coefficients.

In fact, using Maple, we've discovered that for $0 \leq m_1, m_2, m_3 \leq 16$ all such rational functions of u have positive coefficients. The contents of Appendix B are programs used to accomplish this goal.

The program **extract** extracts the rational function of u from ϕ. Then **check** simplifies the result of the substitution $u = 1 - w$, the result being a rational function of w with denominator $w^{M_1+M_2+M_3+1}$. **check** determines if the numerator polynomial has positive coefficients, which if true, implies positivity of infinitely many coefficients of ϕ. Unfortunately, our efforts to describe a pattern to the numerator polynomials have been unsuccessful.

3.2 Gillis and Kleeman's idea

To understand the proof used by Gillis and Kleeman that was mentioned earlier [GK79], let

$$\psi(r,s,t) := \sum D(m_1,m_2,m_3) r^{m_1} s^{m_2} t^{m_3} := \frac{1}{(1-r-s-t+4rst)^{\alpha+1}},$$

$$Q_1(r,s,t) := (1-K_1) - K_2 r + 2K_1 t + 2K_2 rt$$

and

$$Q_2(r,s,t) := K_1 + K_2 r + 2(1-K_1)s - 2K_2 rs,$$

where K_1 and K_2 are arbitrary constants. If we set $b_1 := r(Q_1 + Q_2) - 1$, $b_2 := s(-Q_1 + Q_2)$, $b_3 := t(Q_1 - Q_2)$ and $b_4 := (\alpha+1)(Q_1 + Q_2)$, then

$$b_1(r,s,t)\frac{\partial \psi}{\partial r} + b_2(r,s,t)\frac{\partial \psi}{\partial s} + b_3(r,s,t)\frac{\partial \psi}{\partial t} + b_4(r,s,t)\psi = 0. \qquad (3.2)$$

In other words, since the constant term of b_1 is -1, $\frac{\partial\psi}{\partial r}$ can be added to both sides of (3.2), yielding

$$\begin{aligned}&\frac{\partial\psi}{\partial r}\\ = &\ (Q_1+Q_2)r\frac{\partial\psi}{\partial r}+(-Q_1+Q_2)s\frac{\partial\psi}{\partial s}+(Q_1-Q_2)t\frac{\partial\psi}{\partial t}+(\alpha+1)(Q_1+Q_2)\psi.\end{aligned}$$

This in turn leads to the recurrence relation

$$\begin{aligned}&(m_1+1)D(m_1+1,m_2,m_3)\\ &= (m_1-(1-2K_1)m_2+(1-2K_1)m_3+\alpha+1)D(m_1,m_2,m_3)\\ &+ 2K_2(m_2-m_3)D(m_1-1,m_2,m_3)\\ &+ 2(1-K_1)(m_1+m_2-m_3+\alpha)D(m_1,m_2-1,m_3)\\ &+ 2K_1(m_1-m_2+m_3+\alpha)D(m_1,m_2,m_3-1)\\ &- 2K_2(m_1+m_2-m_3+\alpha-1)D(m_1-1,m_2-1,m_3)\\ &+ 2K_2(m_1-m_2+m_3+\alpha-1)D(m_1-1,m_2,m_3-1). \qquad (3.3)\end{aligned}$$

This is a generalization Gillis and Kleeman's idea which was used to prove positivity of the coefficients of ψ for the case $\alpha=0$.

To retrieve the Gillis and Kleeman recurrence relation,

$$\begin{aligned}&(m_1+1)D(m_1+1,m_2,m_3)\\ = &\ (m_1-m_2+m_3+1)D(m_1,m_2,m_3)+2(m_1-m_2+m_3)D(m_1,m_2-1,m_3) \qquad (3.4)\end{aligned}$$

(which deals with the case $\alpha=0$) set $K_1=1$ and $K_2=0$. Then (3.4) can be used to prove (inductively) that $D(m_1,m_2,m_3)\geq 0$ since all of the coefficients of (3.4) are non-negative. Symmetry of the coefficients of ψ allows the assumption $m_1\geq m_2\geq m_3\geq 0$.

Gillis, Reznick and Zeilberger [GRZ83] used the same idea to prove positivity for $\alpha\geq(-5+\sqrt{17})/2$. The induction argument requires $D(m_1,m_1,1)\geq 0$, which in turn depends upon $\alpha \geq (-5+\sqrt{17})/2$. For good measure, we note that the conditions $0 \leq K_1 \leq 1$ and $K_2 = 0$ lead to a generalized version of Gillis, Reznick and Zeilberger's result; i.e.,

$$\begin{aligned} & (m_1+1)D(m_1+1, m_2, m_3) \\ = & (m_1 + (2K_1 - 1)m_2 - (2K_1 - 1)m_3 + \alpha + 1)D(m_1, m_2, m_3) \\ + & 2K_1(m_1 - m_2 + m_3 + \alpha + 1)D(m_1, m_2, m_3 - 1) \\ + & 2(1 - K_1)(m_1 + m_2 - m_3 + \alpha + 1)D(m_1, m_2 - 1, m_3) \end{aligned}$$

is a recurrence relation with non-negative coefficients for all $0 \leq K_1 \leq 1$.

In summary, the importance of this generalization is that the so-called Gillis and Kleeman recurrence relation (3.4) is only one of many recurrence relations for which the coefficients of ψ are a solution. The same holds for the coefficients of our function ϕ. Among the many recurrences, perhaps there is one for which the coefficients are always non-negative, subject to the condition that $m_1 \geq m_2 \geq m_3 \geq m_4 \geq 0$.

Of course, existence of such a recurrence is a stronger assertion than the one we make. Nonetheless, we've found 24 polynomials $c_{i_1,i_2,i_3,i_4}(r,s,t,u)$, such that

$$\begin{aligned} & (m_1+1)P(m_1+1, m_2, m_3, m_4, m_5) \\ = & \sum c_{i_1,i_2,i_3,i_4}(r,s,t,u)P(m_1 - i_1, m_2 - i_2, m_3 - i_3, m_4, i_4) \end{aligned} \tag{3.5}$$

where $0 \leq i_1 \leq 2$ and $0 \leq i_2, i_3, i_4 \leq 1$. The important fact is that there are 25 free parameters; i.e., coefficients of the polynomials. Maybe they can be chosen so that all of the coefficients of the recurrence are positive; it's quite a large problem.

The programs in the file `GandK` were used to find the polynomials and the contents of the file can be found in Appendix C. Alternatively, one may wish to get a copy of the file via anonymous `ftp` to `ftp.math.temple.edu`.

3.3 Generalizing the conjecture

Szegő's brilliant observation [A75] that

$$(1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)=f'(1),$$

with $f(x) := (x-r)(x-s)(x-t)$, led him to discover the generalization that $[f'(1)]^{-1}$ for any number of variables r_1, r_2, ..., r_n. Borrowing Szegő's notation, we note that $R(r,s,t,u) = f''(1)$ for the four variable case. In addition, for the case of three or fewer variables, it's fairly obvious that $[f''(1)]^{-1}$ has positive power series coefficients. This led us to suspect that $[f''(1)]^{-1}$ has positive McLaurin series coefficients for all n.

We experimented on the computer with Maple, and the results are certainly persuasive for the cases of $n = 5, 6, \ldots$. In fact, we wrote the five variable equivalent of the program **check** to see if it's true that $[f''(1)]^{-1}$ has infinitely many positive coefficients for $n = 5$. The results were conclusive for $0 \leq m_1, m_2, m_3, m_4 \leq 10$.

Let $S_m(n)$ denote the elementary symmetric function on $\{1-r_1, 1-r_2, \ldots, 1-r_n\}$ of order m. In the language of combinatorics, $S_{n-1}(n) := f'(1)$ and $S_{n-2}(n) := f''(1)$. It seems that using elementary symmetric functions of any order, as a denominator, yield rational functions with positive McLaurin series coefficients. Thus, positivity of the coefficients of ϕ seems to be a specific case of a general phenomenon, one for which a specific case is already known to be true; i.e., Szegő's work.

REFERENCES CITED

[A26] Abel, N. 1826. *Beweis eines Ausdruckes, von welchem die Binomial-Formel ein einzelner Fall ist.* Crelle's J. Mathematik **1**: 159-160.

[A75] Askey, Richard. 1975. *Orthogonal Polynomials and Special Functions.* Philadelphia: SIAM.

[AG72] Askey, R. and Gasper, G. 1972. *Certain rational functions whose power series have positive coefficients.* Amer. Math. Monthly **79**: 327-341.

[AIK78] Askey, R., Ismail, M.E.H. and Koornwinder, T. 1978. *Weighted permutation problems and Laguerre polynomials.* J. Combinatorial Theory ser. A **25**: 277-287.

[C74] Comtet, Louis. 1974. *Advanced Combinatorics.* Holland: Dordrecht.

[EM96] Ekhad, S. and Majewicz, J. 1996 *A short, WZ-Style proof of Abel's identity.* Elec. J. Combin. **3(2)**: R16.

[F47] Fasenmeyer, M.C. 1947. *Some generalized hypergeometric polynomials.* Bull. Amer. Math. Soc. **67**: 806-812.

[GK79] Gillis, J. and Kleeman, J. 1979. *A combinatorial proof of a positivity result.* Math. Proc. Camb. Phil. Soc. **86**: 13-19.

[GRZ83] Gillis, J., Reznick, B. and Zeilberger, D. 1983 *On elementary methods in positivity theory.* SIAM J. Math. Anal. **14**: 396-398.

[G78] Gosper, R.W., Jr. 1978 *Decision procedure for indefinite hypergeometric summation.* Proc. Natl. Acad. Sci USA **75**: 40-42.

[GKP89] Graham, R.L., Knuth D.E. and Patashnik O. 1974. *Concrete Mathematics.* Reading, MA: Addison-Wesley.

[IT79] Ismail, M.E.H. and Tamhankar, M.V. 1979. *A combinatorial approach to some positivity problems.* SIAM J. Math. Anal. **10**: 478-485.

[M96] Majewicz, John. 1996. *WZ-Style certification and Sister Celine's technique for Abel-type sums.* J. Diff. Eq. Appl. **2**: 55-65.

[S33] Szegö, Gabor. 1933. *Über gewisse Potenzeihen mit lauter positiven Koeffizienten.* Math. Z. **37**: 674-688.

[WZ92] Wilf, H. and Zeilberger, D. *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities.* Invent. math. **108**: 575-633.

[WZ96] Wilf, H. and Zeilberger, D. 1996. *A=B.* Wellesley, MA: A.K. Peters.

APPENDIX A
THE PROGRAM celine

This Maple program accompanies the paper, "WZ-Style Certification and Sister Celine's Technique for Abel-Type Sums" [M96]. It is also available via anonymous ftp to ftp.math.temple.edu

For a hypergeometric term $F(n,k)$, the program `celine` returns an operator of order ORDER, with coefficients of degree DEG, which annihilates the Abel-type term $F(n,k,r,s) := F(n,k)r(r+k)^{k-1}(n-k+s)^{n-k}$. If ORDER or DEG are too low, `celine` will return a message instructing the user to try a larger value or either DEGREE or ORDER.

```
celine:=proc(F,ORDER,DEG)
local testr, tests, var, EXPR, ope, JOE1, JOE2, eq, temp_eq,
temp_var, temp_ope, opefull, jj, ll, i, j, ii:
testr:=2:  tests:=1:  JOE1:=ORDER: JOE2:=DEG:
var:=a[0,0]:
EXPR:=a[0,0]:  ope:=a[0,0]:

# At this stage of the procedure, celine will do three things.
# First, it will create the set of coefficients for an
# operator ope annihilating the Abel-type term F(n,k,r,s).
# It also creates the operator ope itself, as well
```

```
# as the expression EXPR which, if ''large enough'', will
# have coefficients wrt k coefficients wrt k equal to 0.

for i from 1 to JOE1 do
     for j from 0 to JOE2 do
     var:=var union a[i,j]:
     EXPR:=EXPR+a[i,j]*simplify(subs(n=n-i,k=k-j,F)/F)
     *(r+j)*(n-k+s)^(-i+j)/r/(k+r)^j:
     ope:=ope+a[i,j]*N^i*K^j*S^(-i+j)*R^(-j) :
     od:
od:
EXPR:=normal(EXPR): EXPR:=numer(EXPR): EXPR:=expand(EXPR):

# eq will be the set of coefficients

eq:={}:
for ii from 0 to degree(EXPR,k) do
     eq:=eq union coeff(EXPR,k,ii)=0:
od:

# It's much quicker for Maple to find solutions a[i,j]
# for specific values of r, s and n.  Moreover, if no
# solution exists for specific values of r,s and n, then
a general solution doesn't exist.  Therefore, before
# proceeding, celine checks to see if there is a particular
# solution.  If a'non-zero' solution exists, celine proceeds
# to find the general solution.  Otherwise, it stops.
```

```
temp_eq:=subs(r=testr,s=tests,n=1,eq):
lprint(`temp_q has`,nops(temp_eq),`equations to be solved.`);
temp_var:=solve(temp_eq,var):
temp_ope:=subs(temp_var,ope):
if temp_ope=0 then
     RETURN(`No recurrence of order`,JOE1,`degree`,JOE2,`.  Try
     a larger value for DEGREE or ORDER`);
fi:
var:=solve(eq,var);
ope:=subs(var,ope):
opefull:=ope;
ope:=subs(K=1,opefull):
# If there are infinitely many solutions a[i,j], celine
# returns the one with all such a[i,j]=1.  This is the
# purpose of the next double loop.

for jj from 1 to JOE1 do
     for ll from 0 to JOE2 do
     ope:=subs(a[jj,ll]=1,ope):
     od:
od:
ope;
lprint(ope):
end:
```

APPENDIX B
THE PROGRAM check

The information on the following two pages, when programmed in Maple, allows one to extract from the McLaurin series representation of the function

$$\phi := \frac{1}{R} = \sum_{m_1,m_2,m_3,m_4} P(m_1,m_2,m_3,m_4) r^{m_1} s^{m_2} t^{m_3} u^{m_4},$$

the rational function of u defined by

$$F_{M_1,M_2,M_3}(u) := \sum_{m_4} P(M_1,M_2,M_3) u^{m_4}.$$

The program extract does the job of extracting the rational function F from ϕ. Then the substitution $u = 1 - w$ transforms F into a rational function of w, with denominator equal to $w^{M_1+M_2+M_3+1}$. check determines whether or not the numerator polynomial of w has any negative coefficients.

```
R:=(1-r-s-t-u+(2/3)*(r*s+r*t+r*u+s*t+s*u+t*u)):

phi:=1/R:

vars:=[r,s,t,u]:

extract:=proc(a,b,c)

     local F:

     if a<0 or b<0 or c<0 then RETURN(`values too low`):  fi:

     F:=taylor(phi,r,a+1):

     F:=coeff(F,r,a):
```

```
        F:=taylor(F,s,b+1):
        F:=coeff(F,s,b):
        F:=taylor(F,t,c+1):
        F:=coeff(F,t,c):
        F:=factor(normal(expand(F))):
end:

check:=proc(A,B,C)
        local a,b,c,gu,du,lu,i,j,k,l:
        a:=A: b:=B: c:=C:
        lu:=0:
        for i from 0 to a do
          for j from i to b do
            for k from j to c do
            gu:=extract(i,j,k):
            gu:=subs(u=1-w,gu):
            gu:=expand(numer(gu)):
            for l from 0 to degree(gu,w) do
              du:=coeff(gu,w,l):
              if du<0 then
                lu:=lu+1:
                print(`RATS! I found a negative coefficient`);
                print(du,`when i=`,i,`j=`,j,`and k=`,k);
              fi:
            od:
            od:
```

```
        od:
      od:
if lu=0 then print('ALL COEFFICIENTS ARE NON-NEGATIVE! '):  fi:
end:
```

APPENDIX C
THE PROGRAM GandK

The contents of this appendix are the programs used to find the polynomials P_i, $i = 1, \ldots, 5$, such that

$$\frac{\partial\phi}{\partial r} = (P_1)r\frac{\partial\phi}{\partial r} + (P_2)s\frac{\partial\psi}{\partial s} + (P_3)t\frac{\partial\phi}{\partial t} + (P_4)u\frac{\partial\phi}{\partial u} + (P_5)\phi.$$

The summands of P_i are of the form $b_{i,i1,i2,i3,i4}$. P_i are substituted into the above equation, a common denominator is found and the $b_{i,i1,i2,i3,i4}$ are determined so that the two sides are equal.

Finally, the program called `getit` takes the set of P_i's and returns the right-hand side of the recurrence relation for which the coefficients of ϕ are a solution. Of course, the left-hand side of the recurrence relation is always $(m_1+1)P(m_1+1, m_2, m_3, m_4)$. The coefficients of `P(m1-i1,m2-i2,m3-i3,m4-i4)` for the summands of `rec` are the 24 polynomials mentioned in section 3.2. The set `free` will contain the 25 free parameters.

Via anonymous ftp to ftp.math.temple.edu, one can "get" the file called "GandK". Once read into a Maple session, the programs will give the desired recurrence relation (the right-hand side).

```
print('The purpose of this file is to find polynomials');
print('P.i such that the sum');
print('P.1*rdr+P.2*sds+P.3*tdt+P.4*udu+P.5*R=1-(2/3)*(s+t+u).');
```

```
print('This is accomplished by making');
print('(P.1-1)*rdr+P.2*sds+P.3*tdt+P.4*udu+P.5*R=0');
print('If such P.i's exist, then perhaps the free parameters');
print('can be chosen so that the coefficients of phi');
print('satisfy a linear recurrence with positive coefficients.');

# The idea of the programs in this file is to form
# the numerator polynomial of
#
# (P.1*r-1)*diff(phi,r)+P.2*s*diff(phi,s)+
#             P.3*t*diff(phi,t)+P.4*u*diff(phi,u)+P.5*phi
#
# and form a set of equations from the coefficients of
# r^i*s^j*t^l*u^l.
# The variables in the set of max(i)*max(j)*max(k)*max(l)
# equations are the coefficients of the polynomials P.i.
# There may be a solution if the degree of each P.i
# is chosen so that the number of variables exceeds the
# the number of equatons.  Thus, for example, if
# 0 le i,j,k,l le 1, for each P.1, P.2, P.3, P.4 and P.5,
# then each P.i starts off with 16 summands and
# so there are 80 variables all together.

# The program coe extracts the coefficient of
# (r^i)*(s^j)*(t^k)*(u^l) from a polynomial.

coe:=proc(pol,i,j,k,l)
```

```
      local poly,lu,ir,is,it,iu:
      poly:=pol:  ir:=i:  is:=j:  it:=k:  iu:=l:
      lu:=coeff(coeff(coeff(coeff(poly,r,ir),s,is),t,it),u,iu):
end:

phi:=1/(1-r-s-t-u+(2/3)*(r*s+r*t+r*u+s*t+s*u+t*u)):
R:=1-r-s-t-u+(2/3)*(r*s+r*t+r*u+s*t+s*u+t*u):

# dr, ds, dt and du represent the numerator polynomials
# of the rational functions
# diff(phi,r)/R^2, diff(phi,s)/R^2, etc.

dr:=R^2*diff(phi,r):
ds:=R^2*diff(phi,s):
dt:=R^2*diff(phi,t):
du:=R^2*diff(phi,u):

# Build up the generic polynomials P.i to be used
# the formation of monster (the numerator polynomial) and keep
# building up the set of variables.  Afterwards, form monster.

var:={}:
for i from 1 to 5 do
P.i:=0:
     for i1 from 0 to 2 do
       for i2 from 0 to 1 do
         for i3 from 0 to 1 do
           for i4 from 0 to 1 do
```

```
P.i:=P.i+(b.i.i1.i2.i3.i4)*r^i1*s^i2*t^i3*u^i4:
var:=var union b.i.i1.i2.i3.i4:
od:
od:
od:
od:
od:
mon.1:=(P.1*r-1)*dr:
mon.2:=P.2*s*ds:
mon.3:=P.3*t*dt:
mon.4:=P.4*u*du:
mon.5:=P.5*R:
premonster:=mon.1+mon.2+mon.3+mon.4+mon.5:
monster:=expand(premonster):

# If the coefficient of r^i*s^j*t^l*u^l is non-numeric,
# then set it equal to zero and put it into the
# set of equations.

eq:={}:
for i1 from 0 to degree(monster,r) do
     for i2 from 0 to degree(monster,s) do
       for i3 from 0 to degree(monster,t) do
         for i4 from 0 to degree(monster,u) do
           pu:=coe(monster,i1,i2,i3,i4):
           if type(pu,numeric) then
```

```
          eq:=eq:
        else
          eq:=eq union pu:
        fi:
      od;
    od;
  od;
od;
print(`Now I'm trying to solve a system of`,nops(eq),`equations`);
print(`with`,nops(var),`unknowns¡):
sol:=solve(eq,var):
if sol=NULL then
     print(`No solution; sorry¡);
else
     premonster:=subs(sol,premonster):
fi:

# Some of the coefficients of P.i will need to be numeric to
# accomplish the goal; others may not.  'free' keeps track of
# those that are non-numeric.

free:={}:
for i from 1 to nops(sol) do
     if op(1,op(i,sol))-op(2,op(i,sol))=0 then
       free:=free union op(1,op(i,sol)):
     fi:
```

```
od:
for i from 1 to 5 do P.i:=subs(sol,P.i):od:

# 'getit' is a program, which when given the set of five
# polynomials, returns the linear recurrence relation for
# which the coefficients P(m1,m2,m3,m4) are a solution.

getit:=proc(p1,p2,p3,p4,p5)
local Q,p,rec,i1,i2,i3,i4,i,gu:
Q.1:=m1:
Q.2:=m2:
Q.3:=m3:
Q.4:=m4:
Q.5:=1:
p.1:=p1:  p.2:=p2:  p.3:=p3:  p.4:=p4:  p.5:=p5:
rec:=0:
for i1 from 0 to 2 do
     for i2 from 0 to 1 do
       for i3 from 0 to 1 do
         for i4 from 0 to 1 do
           gu:=0:
           for i from 1 to 5 do
           gu:=gu+coe(p.i,i1,i2,i3,i4)*
           subs(m1=m1-i1,m2=m2-i2,m3=m3-i3,m4=m4-i4,Q.i):
           od:
           gu:=simplify(normal(expand(gu))):
```

```
          rec:=rec+gu*P(m1-i1,m2-i2,m3-i3,m4-i4):
        od;
      od;
    od;
od;
RETURN(rec):
end:
rec:=getit(P.1,P.2,P.3,P.4,P.5);
```