An experimental mathematics approach to some combinatorial problems

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Overview

- Introduction
- Random walks
- Simultaneous core partitions
- Inclusion-exclusion
- Boolean functions
Introduction: experimental mathematics

- Computer as an essential tool, not just a fancy calculator
- Symbolic computing power (Maple in our case)
- Central ideas:
  - ansätze (“guess and check”)
  - generating function methods
  - dynamical programming
  - OEIS
  - distributions of combinatorial statistics
Random walks

- Gambler starts with $0.
- Wins or loses $1 after each round, with equal probability.
- Finitely many steps.
Walk statistics

- Feller [5] defines the following statistics of such a walk $w$:
  - $l(w) = \text{length}$
  - $a_1(w) = \# \text{ of losing times (when money < 0)}$
  - $a_2(w) = \# \text{ of break-even times (money = 0)}$
  - $a_3(w) = \text{time of last break-even}$
  - $a_4(w) = \# \text{ of sign changes}$.

- **Question**: how are the statistics distributed?
Related problem

- Consider walks in plane from \((0, 0) \rightarrow (a, b)\).
- Each step one unit right (losing) or up (winning)
- Breaking even if \(y = x\); losing region is \(y < x\).
Moments

Let $X$ be a random variable. Recall the following:

- **$k$th (straight) moment** $= \mathbb{E}[X^k]$ ($= \mu$ if $k = 1$)
- **$k$th central moment** $= \mathbb{E}[(X - \mu)^k]$ ($= \sigma^2$ if $k = 2$)
- **$k$th standardized moment** $= \mathbb{E}[(X - \mu)^k]/\sigma^k$
- Standardized moments = “fingerprint” of distribution
- E.g., $N(0, 1)$ has standardized moments $0, 1, 0, 3, 0, 15, 0, 105, \ldots$ (A123023).
Big, important questions

Given an indexed random variable $X_n$ of a certain class of combinatorial objects (e.g., walks),

- Express the moments in terms of $n$.
- Investigate the asymptotic distribution. Do the standardized moments approach some recognized sequence as $n \rightarrow \infty$?
Moments from generating functions

- Suppose $X$ has finite sample space $S$, all outcomes equally likely.
- Define the g.f. 
  \[ f(t) = \sum_{a \in S} t^{X(a)}, \]

  a finite polynomial in $t$.
- $k$th straight moment is
  \[ \left( t \frac{d}{dt} \right)^k f(1) \frac{1}{|S|}. \]
Our results for Feller’s problem

- We utilized Dr. Z.’s existing Feller package to derive results about the moments. Maple’s “convert to formal power series” function was useful.
- For example, some information for the statistic “# of visits to $y = x$ of a uniform random walk from $(0, 0) \rightarrow (n, n)$” are shown on the next slide.
Mean:

\[ \mu = \frac{-(2n)! + 4^n (n!)^2}{(2n)!} \]

Variance:

\[ \sigma^2 = -\frac{16^n (n!)^4 + 4^n (n!)^2 (2n)! - 4n ((2n)!)^2 - 2 ((2n)!)^2}{((2n)!)^2} \]

Limits of 3rd-5th standardized moments:

\[ 2 \frac{\sqrt{\pi} (\pi - 3)}{(-\pi + 4)^{3/2}}, -\frac{3 \pi^2 - 32}{\pi^2 - 8\pi + 16}, 4 \frac{\sqrt{\pi} (\pi^2 + 5\pi - 25)}{(-\pi + 4)^{5/2}} \]
Arbitrary step sets

- Fix $S \subset \mathbb{N}^2$, $|S| < \infty$.
- Let $W_{a,b}^S = \{\text{walks from } (0, 0) \to (a, b) \text{ with steps in } S\}$.
- How do the statistics of a uniform random $w \in W_{n,n}^S$ behave?
- Can we get asymptotic estimates of the moments as $n \to \infty$?
Dynamic programming scheme

\[ W_{a,b}^S = \bigcup_{s \in S} W_{(a,b)-s} \{s\} \]
Computing generating functions

- For fixed $S, a, b$, the g.f.
  \[ F_{a,b}(t) := \sum_{w \in W_{a,b}^S} t^{a_1(w)} \]
  is a \textit{finite} polynomial in $t$.

- $F_{a,b}(t)$ can efficiently computed using the DP scheme above
  and option remember!

- Generate moment data from $F_{n,n}(t)$ for many values of $n$, and
  numerically analyze moment asymptotics.
## Sample of the storybook

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<thead>
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<th>Steps</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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Walks in three dimensions

- Previously: winning ($y > x$), losing ($y < x$), and break-even ($y = x$) regions
- Now, 7 regions: $x < y < z$, ..., $z < y < x$, and “none of the above”
- We implemented analogous generating functions for statistics tracking # of visits to each region.
- A lot slower.
Partitions

- **Partition of** $n \in \mathbb{N}$: a nonincreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_i \in \mathbb{N}$ and $\sum \lambda_i = n$.
- $n = |\lambda|$ is the **size**.
- $\lambda_1, \ldots, \lambda_k$ are the **parts**.
- E.g., $9 = 4 + 3 + 1 + 1$.
- Appear in representation theory, statistical mechanics, etc.
Young diagram of \((4, 3, 1, 1)\)
Hook length of a cell
Young diagram of $(4, 3, 1, 1)$ showing hook lengths
Core partitions

- A partition is an $s$-core if it avoids hook length $s$ (some definitions equivalently say “divisible by $s$” [9]).
- A (simultaneous) $(s, t)$-core avoids both hook lengths $s$ and $t$.


The number of $(s, t)$-cores is finite iff $\gcd(s, t) = 1$, in which case it is

$$(s + t - 1)!/(s!t!).$$

- Catalan (A000108) if $t = s + 1$. 

Distribution of size

- If $\gcd(s, t) = 1$, consider random variable “size of a u.r. $(s, t)$-core.”

Theorem (Conjectured by Armstrong [2], proved by Johnson [7])

*The average size of an $(s, t)$-core partition is*

$$\frac{(s - 1)(t - 1)(s + t + 1)}{24}.$$  

- Using Maple, Dr. Z. went up to the 6th moment.
Distinct parts

- Analysis seems harder if we require distinct parts.

**Theorem (Straub 2016 [10])**

*The number of \((s, s + 1)\)-cores with distinct parts is \(F_{s+1}\).*

- Size distribution?
Our work: size of an \((s, s + 1)\)-core with distinct parts

- Define \(P_s = \{(s, s + 1)\text{-cores with distinct parts}\}
- G.f. of size:
  \[
  G_s(q) := \sum_{p \in P_s} q^{|p|}
  \]
- Need a fast way to compute \(G_s(q)\).
Straub’s characterization of $p \in P_s$

- Perimeter $< s$
- Distinct parts
Recursive scheme for g.f.

- Perimeter formulation allows us to express the g.f. as a $q$-binomial sum:

\[ G_s(q) = \sum_{m=0}^{s} q^{\frac{m+1}{2}} \binom{s-m}{m}_q. \]

- Use Dr. Z.’s package qEKHAD to find and prove a recursion for efficient computation.

- Also implies that the moments must satisfy the $C$-finite ansatz!
Sample results

Let $X_s =$ “size of a u.r. $(s, s + 1)$-core with distinct parts.”

Theorem

$$\mathbb{E}[X_s] = \frac{1}{50} \frac{5 s^2 F_{s+1} - 6 s F_s + 7 s F_{s+1} - 6 F_s}{F_{s+1}}.$$ 

- We got up to moment 16.
- Standardized moments approach 0, 1, 0, 3, 0, 15, 0, 105, ... 
- Conjecture: $X_s$ is asymptotically normally distributed! (Contrast with previous case)
(2s + 1, 2s + 3)-cores with distinct parts

- Here, less lucky
- Use Anderson’s bijection
- Define poset $P_{s,t} := \mathbb{N}\setminus(s\mathbb{N} + t\mathbb{N})$, with $c \leq_P d \iff d - c = n_1s + n_2t$
- Order ideal: subset closed under $\leq_P$
- $(s, t)$-cores $\leftrightarrow$ order ideals of $P_{s,t}$. 
The poset $P_{9,10}$ and an order ideal (Catalan decomposition)
Illustrating the correspondence

\[
\begin{array}{ccc}
7 & & \\
5 & & \\
2 & & \\
1 & & \\
\end{array}
\leftrightarrow \{1, 2, 5, 7\}
\]

distinct parts = no consecutive labels
Core partitions

$(2s + 1, 2s + 3)$-cores with distinct parts

$P_{2s+1,2s+3}$ for $s = 6$

- $11$
- $24 \ 9$
- $37 \ 22 \ 7$
- $50 \ 35 \ 20 \ 5$
- $63 \ 48 \ 33 \ 18 \ 3$
- $76 \ 61 \ 46 \ 31 \ 16 \ 1$
- $89 \ 74 \ 59 \ 44 \ 29 \ 14$
- $102 \ 87 \ 72 \ 57 \ 42 \ 27 \ 12$
- $115 \ 100 \ 85 \ 70 \ 55 \ 40 \ 25 \ 10$
- $128 \ 113 \ 98 \ 83 \ 68 \ 53 \ 38 \ 23 \ 8$
- $141 \ 126 \ 111 \ 96 \ 81 \ 66 \ 51 \ 36 \ 21 \ 6$
- $154 \ 139 \ 124 \ 109 \ 94 \ 79 \ 64 \ 49 \ 34 \ 19 \ 4$
- $167 \ 152 \ 137 \ 122 \ 107 \ 92 \ 77 \ 62 \ 47 \ 32 \ 17 \ 2$
Look at the smallest vacant odd label

- 11
- 24  9
- 37  22  7

- 3
- 16  1
- 29  14
- 42  27  12
- 55  40  25  10
- 68  53  38  23  8
- 81  66  51  36  21  6
Computing the g.f. and moments

- Decomposition gives a (complicated) recursive scheme for the g.f.

**Theorem**

The average size of a \((2s + 1, 2s + 3)\)-core partition with distinct parts is

\[
\frac{1}{32}(10 s^3 + 27 s^2 + 19 s).
\]

- Expressed moments up to 7th as polynomials in \(s\)
- Not asymptotically normal.
Other families of cores

- We can play the poset game some more.
- For each class of cores, we get a class of posets.
- Recursively characterize the order ideals
- Develop a scheme to compute g.f.s and find moments.
- We also got results for \((s, ds - 1)\)-cores with distinct parts, \((s, s + 1)\)-cores with parts repeated \(\leq k\) times, and \((s, s + 1)\)-cores with odd parts (Johnson responded with an elegant abacus approach [8] and Dr. Z. donated $200 to the OEIS.)
Inclusion-exclusion

**Theorem (principle of inclusion-exclusion)**

Let $A_1, \ldots, A_N$ be events in a finite probability space. For $I \subset [N]$, define

$$A_I = \bigcap_{j \in I} A_j.$$ 

Then,

$$\Pr \left[ \bigcup_i A_i \right] = \sum_{i=1}^{N} (-1)^{i+1} \sum_{I \subset [N], |I| = i} \Pr[A_I].$$
Thesis Defense
Inclusion-exclusion

 PIE and the probabilistic method

- Truncating after first term gives

\[ \Pr \left[ \bigcup_i A_i \right] \leq \sum_i \Pr[A_i]. \]

- Boole’s inequality, often used in probabilistic method.
- Idea: compute more terms in the sum before truncating for better bound (Bonferroni inequalities).
Improving the Erdős lower bound on $R(k, k)$
Boolean satisfiability

- $n$ boolean variables $x_1, \ldots, x_n$
- *Conjunctive normal form (CNF)*: e.g.,

  $\neg x_3 \land (x_2 \lor x_3) \land (x_1 \lor \neg x_2)$

  (Dual: *disjunctive normal form (DNF)*)
- SAT: given a CNF, determine whether it is satisfiable
- NP-complete.
Consider dual of SAT: determine whether a DNF $C_1 \lor \cdots \lor C_N$ is a tautology.

Randomly assign to $x_1, \ldots, x_n$ and let $A_i$ be the event $C_i = \text{True}$.

Tautology iff $\Pr[\bigcup_i A_i] = 1$

Use Bonferroni bounds!

Our solver is not competitive, but maybe theoretically interesting.
Covering systems (Erdős 1950 [4])

- **Covering system**: finite set of congruences
  \[ \{ a_i \pmod{m_i} : 1 \leq i \leq N \} \]
  whose union is \( \mathbb{N} \)

- **Exact covering**: disjoint congruences. E.g.,
  \[ \{ 0 \pmod{2}, 1 \pmod{2} \} \]

- **Distinct**: distinct moduli. E.g.,
  \[ \{ 0 \pmod{2}, 0 \pmod{3}, 1 \pmod{4}, 5 \pmod{6}, 7 \pmod{12} \} \].
Another exact covering

\{0 \text{ (mod 2)}, 1 \text{ (mod 4)}, 3 \text{ (mod 4)}\}
Covering system facts

- A C.S. cannot be both exact and distinct. (Mirsky-Newman/Davenport-Rado)
- Erdős asked whether the smallest modulus $m_1$ of a distinct C.S. can be arbitrarily large. In 2015, Hough [6] proved $m_1 < 10^6$.
- Via the Chinese remainder theorem, Berger and Felzenbaum [3] described C.S. as a covering of a finite $p_1 \times \ldots p_k$ box with sub-boxes.
- Exact $= \text{disjoint sub-boxes}$
- Distinct $= \text{non-parallel sub-boxes}$.
Our work: Boolean analogs

- We can view a DNF tautology $C_1 \lor \cdots \lor C_k$ in $x_1, \ldots, x_n$ as a covering of the Boolean $n$-cube by sub-cubes corresponding to the clauses ("congruences").
- Define the support ("modulus") of a clause $C_i$ as the participating variables; e.g. $x_1 \land \neg x_2$ has support $\{x_1, x_2\}$.
- *Exact*: disjoint sub-cubes (any two clauses conflict)
- *Distinct*: clauses supported on distinct sets.
Minimum clause size in a distinct DNF tautology

- Unlike the Erdős case, minimum clause size $k$ is unbounded (easy $k = n/2$ construction). For each $n$, how large can we get?
- Density argument:

\[
\sum_{i=k}^{n} \binom{n}{i} \frac{1}{2^i} \geq 1
\]

gives rise to upper bound $k \leq A_n = 1, 1, 1, 2, 3, 4, 5, 5, \ldots$

- Using computer search methods, we constructed optimal DNF tautologies for $n \leq 14$ except $n = 10$ (1008/1024 vertices covered).
Constructing uniform distinct DNF tautologies

- What if we instead require all clauses to have same size, $k$?
- Density argument:
  \[
  \binom{n}{k} \frac{1}{2^k} \geq 1,
  \]
  gives rise to upper bound $k \leq A_n = 1, 1, 1, 2, 3, 4, 5, 5, \ldots$
- Using computer search methods, we constructed optimal DNF tautologies for $n \leq 14$ except $n = 3, 5, 9, 13$. 
Thank you!
References I

Partitions which are simultaneously $t_1$ and $t_2$-core.  

Results and conjectures on simultaneous core partitions.  

A nonanalytic proof of the Newman-Znam result for disjoint covering systems.  
References II

On integers of the form $2^k + p$ and some related problems.

*An introduction to probability theory and its application.*

Solution of the minimum modulus problem for covering systems.
References III

Lattice points and simultaneous core partitions.

Simultaneous cores with restrictions and a question of Zaleski and Zeilberger.

The Catalan case of Armstrong’s conjecture on simultaneous core partitions.
References IV

Core partitions into distinct parts and an analog of Euler’s theorem.

Fully AUTOMATED computerized redux of Feller’s (v.1) Ch. III (and much more!).
The End