AN EXPERIMENTAL MATHEMATICS APPROACH TO SOME COMBINATORIAL PROBLEMS

by

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A dissertation submitted to the

School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Doctor Zeilberger

And approved by

New Brunswick, New Jersey

May, 2018

ABSTRACT OF THE DISSERTATION

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While computers have long been used for *numeric* computations, their growing power to handle *symbolic* manipulations is becoming increasingly useful in mathematics. Our "experimental mathematics" approach uses symbolic computing as an essential tool to both conjecture and prove new results, often with little or no human intervention. Here, we will illustrate how we used experimental mathematics to explore several combinatorial problems. Namely, we will start out with a brief analysis of the generating functions of some statistics associated with random walks in the plane. Then, we will do the same for certain families of *simultaneous core* integer partitions; this constitutes the bulk of the thesis and contains our main results. We will briefly cover our attempts to apply computer implementations of inclusion-exclusion to Ramsey theory and Boolean satisfiability. Finally, we will introduce a Boolean analog of Erdős' integer covering systems and go over some related results and conjectures.

Acknowledgements

I thank my advisor, Doron Zeilberger, for directing my research at Rutgers. Dr. Z. introduced me to lots of interesting problems and was always available to offer guidance. He really cares about his students.

I also thank all the the graduate students who were friendly and helpful: Sjuvon Chung, Nathan Fox, Cole Franks, Andrew Lohr, Katie McKeon, Tim Naumovitz, and Matthew Russell, to name a few.

I am grateful to many professors at Rutgers: Zheng-Chao Han, for giving me advice in my first year; Michael Saks, for teaching me combinatorics and serving on my oral committe; Michael Kiessling and Vladimir Retakh, for serving on my oral and defense committees; Neil Sloane, for proof-reading my paper and his talks on integer sequences. I thank Andrew Sills for being my external committee member and proof-reading the draft of this thesis.

Finally, I thank my family—especially my mother and father, who have always been very supportive in all aspects of life.

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Chapter 1

Introduction

The use of computers in mathematics is becoming increasingly prevalent. However, computers are often used merely as calculators, crunching floating-point numbers and computing numerical results. Symbolic computing packages (Maple, for example) make it possible to employ computers for more meaningful work. Often, computers can be programmed to conjecture and provide rigorous proof of new results with little or no human input. Further, the work done by the computer may be so involved that it would be virtually impossible to do by hand. The computer is no longer a secondary tool; it becomes an indispensable co-author. This is the gist of *experimental mathematics* [4]. Here, we will use experimental mathematics to approach several areas of combinatorics: random walks, simultaneous core partitions, Ramsey numbers, and Boolean functions. An overall goal is to illustrate how our experimental mathematics methods proved fruitful in each of these diverse areas.

Note that some of the following material is adapted from our papers published in journals and/or the arXiv. Where this is the case, it is noted at the beginning of the chapter or section.

Also, the supporting Maple packages and computer output referenced throughout this thesis are listed in Appendix A. Also, see Appendix B for an index of notation.

1.1 Random Walks

In Chapter 2, we will discuss Feller's coin tossing experiment [13]. A fair coin is repeatedly tossed, and a dollar is won or lost depending on each outcome. The accumulated winnings can be seen as a random walk in the plane, where steps one unit up or right are allowed. At the nth time step, we can define various statistics for the walk: the number of previous time steps at which we were losing (had negative money) or the number of times we were breaking even, for example. We will use the computer to derive explicit formulas for the moments of these random variables in terms of n. We will also explain how to use dynamic programming techniques to analyze the asymptotics of the moments when arbitrary steps are allowed or higher dimensions are involved.

1.2 Simultaneous Core Partitions

Next, in Chapter 3, we will cover the topic of simultaneous core partitions. A *partition* is a way to break up a positive integer as a nonincreasing sum of positive integers, called the *parts*. For example, $9 = 4 + 3 + 1 + 1$ is way to partition 9, which we call the *size* of this partition. Partitions are prevalent in number theory, representation theory, and statistical mechanics [1].

A partition can be graphically represented by a *Young diagram*, a left-justified arrangement of boxes where the number of boxes in the kth row from the top is the kth part of the partition (see Figure 1.1). Each box has an associated *hook*, which consists of the box itself together with those to its right and below it, and *hook length*, which is the number of boxes comprising the hook. A partition is an s-*core* if its Young diagram has no hooks of size s; it is an (s,t)-*core* if neither s nor t appear as hook lengths in its young diagram [for example, the partition in Figure 1.1 is a 6-core and an 8-core but not a $(6,7)$-core]. It is the latter *simultaneous core* partitions that we will study here.

It turns out that the number of (s,t)-core partitions is finite iff $\gcd(s,t) = 1$. For example, for each s, there are finitely many $(s, s+1)$-core partitions. Straub [22] proved that the number of such partitions is enumerated by the Fibonacci sequence. We will expand on his results, by considering the random variable which is the *size* of such a partition chosen uniformly at random. We will use the computer to compute explicit formulas for the moments of this random variable in terms of s and analyze the limiting distribution as $s \to \infty$. Then, we will conduct similar analyses of other families of core partitions and investigate what happens if only odd parts are allowed.

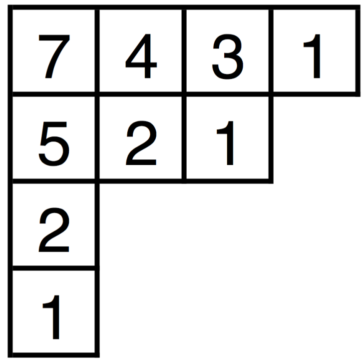

Figure 1.1: Young diagram of the partition $9 = 4 + 3 + 1 + 1$, showing the hook lengths of each box.

1.3 Inclusion-Exclusion

In Chapter 4, we will switch gears to introduce an idea we had for solving some combinatorial problems. The rationale is to implement the inclusion-exclusion formula—which is often truncated and used to bound certain probabilities—with a computer, enabling us to tighten the bounds by computing more terms in the sum. This chapter represents some "long-shot" attempts at big problems, so we include it with the disclaimer that our results are not as promising as in other chapters.

Our inspiration for a computer implementation of inclusion-exclusion came from the problem of bounding diagonal Ramsey numbers. The famous Erdős-Szekeres lower bound uses a simple inclusion-exclusion based bound to prove existence of a coloring of K_n with no monochromatic r-clique [21]. Our idea was to improve this bound by adding more terms of the inclusion-exclusion sum.

We will use a similar technique to approach Boolean satisfiability (SAT). Basically, the problem of SAT is as follows: given a certain Boolean expression f in n variables $x_1, \ldots, x_n$, determine whether there is an assignment to the variables that makes f evaluate to True (or 1). For example, $(x_1 \vee x_2) \wedge \bar{x}_3$ is satisfiable; for example, set $x_1 = 1, x_2 = 0$. On the other hand, $x_1 \wedge \bar{x}_1$ is clearly not satisfiable.

SAT was the first problem proved to be NP-complete [8]; there are no known polynomial time algorithms to find whether an arbitrary Boolean function is satisfiable. We will illustrate how inclusion-exclusion can be used to bound the probability of satisfaction, and give some concrete statistics derived from random input.

1.4 A Boolean Analog of Erdős Covering Systems

The dual of SAT—finding whether a Boolean function is a tautology—is equivalent to checking whether a set of *sub-cubes* of the Boolean cube form a covering. Formulated this way, a tautology is analogous to a *covering system* of the integers by congruence classes. Such coverings were introduced by Erdős [11]. In the final chapter of this thesis, we will adapt certain notions from integer covering systems (for example, *distinct* and *exact* coverings) to the case of Boolean functions, and introduce some new results and conjectures.

Chapter 2

Random Walks

This chapter is adapted from the article [29].

2.1 Introduction: Feller's Coin Tossing Statistics

2.1.1 Coin Tossing and Walks in the Plane

In [13], Feller considers the following experiment. Suppose a gambler tosses a coin finitely many times, winning a dollar whenever heads comes up and losing a dollar when tails appears. The evolution of the game can be described by a string $w = w_1 w_2 \cdots w_n$, where $w_i \in \{-1, 1\}$ describes the outcome of the i^{th} coin toss.

By making the association $-1 = r$ (a right step) and $1 = u$ (an up step), we can also interpret a string w as a walk in $\mathbb{N}^2$ starting from $(0,0)$. For example, the string ur represents a game in which a dollar is won and then lost. Equivalently, ur is a walk from $(0,0) \to (1,0) \to (1,1)$. We shall use the terms "game," "string," and "walk" interchangeably.

Definition 2.1. *Let W be the set of all such walks. We have the following walk statistics (functions from $W \to \mathbb{N}$):*

- *the length (number of steps), $l(w)$;*
- *the number of losing times (points where the walk is below $y = x$):*

$$a_1(w) := \left| \left\{ i : \sum_{j=1}^{i} w_j < 0 \ or \ \left(\sum_{j=1}^{i} w_j = 0 \ and \ \sum_{j=1}^{i-1} w_j < 0 \right) \right\} \right|;$$

- *the number of break-even times (points on $y = x$):*

$$a_2(w) := \left| \left\{ i : \sum_{j=1}^{i} w_i = 0 \right\} \right|;$$

- *the last break-even time:*

$$a_3(w) := \max \left\{ i : \sum_{j=1}^{i} w_j = 0 \text{ } and \text{ } \sum_{j=1}^{i} w_j > 0 \text{ } for \text{ } r > i \right\};$$

- *and the number of sign-changes (points where the walk crosses $y = x$):*

$$a_4(w) := \left| \left\{ i : \sum_{j=1}^{i-1} w_j \cdot \sum_{j=1}^{i+1} w_j < 0 \right\} \right|.$$

Further, we define W_n to be the set of n-step walks in W, and $W_{n,n}$ to be the walks to (n, n).

2.1.2 Old Results

Theorem III.4.1 in [13] states that there are $\binom{2k}{k}\binom{2n-2k}{n-k}$ walks $w \in W_n$ satisfying $a_1(w) = 2k$. For n large and fixed, the distribution of a_1 resembles (modulo scaling) $1/\sqrt{x(1-x)}$, so it is sometimes called the *discrete arcsine* distribution. It is u-shaped, meaning that, surprisingly (or not surprisingly, if you believe in luck), most walks are either winning for most of their duration or losing on the majority of flips.

Theorem III.9 in [13], the Chung-Feller theorem, says that the number of walks to (n, n) with $2k$ losing times is given by a Catalan number, and is independent of k. In terms of generating functions:

$$\sum_{w \in W_{n,n}} t^{a_1}(w) = \frac{1}{n-1}\binom{2n}{n}\sum_{k=0}^{n} t^{2k}.$$

Finally, in [37], Zeilberger uses Maple to evaluate the "grand generating function"

$$\sum_{w \in W} z^{l(w)} t_1^{a_1(w)} t_2^{a_2(w)} t_3^{a_3(w)} t_4^{a_4(w)}$$

as a (very messy) algebraic function of z, t_1, t_2, t_3, t_4.

These results are illuminating for this problem, but they are gotten through ad hoc methods. So, for example, it is not obvious how to derive analogous results for walks in higher dimensions, or walks where non-standard steps are allowed.

Here, we shall give an alternative method to analyze the statistics of a very general class of walks and approximate the long-run behavior of their moments. But first, let us see if we can discover a few more exact results using the analytical approach.

2.2 Exact Results

2.2.1 Moments of Up-Right Walks to (n, n)

Suppose we uniformly randomly pick a walk $w \in W_{n,n}$. Then we can think of $a_1(w)$ as a random variable. For each n, $a_1|W_{n,n}$ ($|$ = "with sample space") has a certain distribution, so it is natural to wonder about the limiting distribution as $n \to \infty$. For example, is it asymptotically normal?

Recall that the kth *straight moment* of a random variable X is given by

$$\mathbb{E}[X^k].$$

For example, when $k = 1$, we get the mean or expected value, μ. Further, the kth *central moment*, or moment about the mean, is

$$\mathbb{E}[(X-\mu)^k].$$

For example, the second central moment is the variance, σ^2. Finally, the kth *standardized* (central) moment is the ratio of the central moment with the kth power of the standard deviation:

$$\frac{\mathbb{E}[(X-\mu)^k]}{\sigma^k}.$$

The sequence of standardized moments of a random variable are a fingerprint of its distribution; for example, the standard normal distribution has moments $0, 1, 0, 3, 0, 15, 0, 105, \ldots$.

In the case of $a_1|W_{n,n}$, we can find the moments in terms of n, which is not surprising since a_1 is essentially uniform by the Chung-Feller rule.

Using the procedure `ChungFeller` in [37], we find

$$\sum_{w \in W_{n,n},\, n \in \mathbb{N}} z^n t^{a_1(w)} = \frac{2}{\sqrt{-4\,z+1} + \sqrt{-4\,zt^2+1}}.$$

Now we use `convert(%,FPS,z)` to convert this function to a formal power series in z. By looking at the coefficient of z^n, we obtain the generating function

$$F_n(t) = \sum_{w \in W_{n,n}} t^{a_1(w)},$$

as a function of n! To see it for yourself, use `ChungFellerGF(t,n)` in the Maple package (keyword `Feller` in Appendix A).

From this generating function, it is easy to compute the moments as functions of n; repeatedly apply the operator $t\partial_t$, then substitute $t = 1$, and normalize by the size of the sample space. This gives the straight moments, from which one can obtain the central and standard moments. This is done in `ChungFellerMoment`. You can easily verify the following:

Proposition 2.2. *The number of losing times of a walk chosen uniformly randomly from $W_{n,n}$ has mean n and variance $n^2/3+2n/3$, and its third through tenth standardized moments about the mean approach $0, 9/5, 0, 27/7, 0, 9, 0, 243/11$ as $n \to \infty$.*

Analogously, for a_2, the number of visits to the diagonal $y = x$, we have

$$F(z,t) := \sum_{w \in W_{n,n},\, n \in \mathbb{N}} z^n t^{a_2(w)} = \frac{1}{t\sqrt{-4\,z+1} - t + 1}.$$

Unfortunately, Maple cannot convert this to a formal power series in z. However, $F_t(z,1)$ *is* convertible to a formal power series, so we can compute $\mathbb{E}[a_2|W_{n,n}] = [z^n]F_t(z,1)/\binom{2n}{n}$, as a function of n. In a similar way, we can find higher moments: the idea is to repeatedly apply the operator $t\partial_t$, substitute $t = 1$, and *then* expand as a formal power series in z.

The moments of a_2 are surprisingly complicated in comparison with those of a_1:

Proposition 2.3. *The number of visits to $y = x$ of a walk chosen uniformly randomly from $W_{n,n}$ has mean and variance*

$$\frac{-\,(2\,n)! + 4^n\,(n!)^2}{(2\,n)!}, -\frac{16^n\,(n!)^4 + 4^n\,(n!)^2\,(2\,n)! - 4\,n\,((2\,n)!)^2 - 2\,((2\,n)!)^2}{((2\,n)!)^2},$$

and its third through fifth standardized moments about the mean approach

$$2\,\frac{\sqrt{\pi}\,(\pi-3)}{(-\pi+4)^{3/2}}, -\frac{3\,\pi^2-32}{\pi^2-8\,\pi+16}, 4\,\frac{\sqrt{\pi}\,\left(\pi^2+5\,\pi-25\right)}{(-\pi+4)^{5/2}}$$

as $n \to \infty$.

2.2.2 Forward King Walks

Now we examine another special set of walks:

Definition 2.4. *Let $K_{n,n}$ be the set of walks from $(0,0) \to (n,n)$ with steps in $\{r,u,d\} = \{(1,0),(0,1),(1,1)\}$. Let $K := \bigcup_{n\in\mathbb{N}} K_{n,n}$. For $w \in K$, let $n(w)$ be the n such that $w \in K_{n,n}$.*

Think of K as the set of journeys possible for a forward-marching King that end on the line $y = x$. Each move, we take a step from $\{r,u,d\}$ (right, up, or diagonal).

For a set $E \subset K$, define the generating function

$$F_E(z,t) := \sum_{w\in E} z^{n(w)} t^{a_1(w)}.$$

We we shall find an algebraic expression for $F_K(z,t)$. The idea is to convert facts describing walks in K to equations involving generating functions.

Definition 2.5.

- *Juxtaposition of two sets A, B of walks denotes concatenation:*

$$AB := \{w_1 w_2 : w_1 \in A, w_2 \in B\}.$$

  If A or B is a singleton, we drop the braces: e.g., $aB := \{a\}B$ for a walk a.

- *The* Kleene star *of a set of walks is its closure under concatenation:*

$$E^* := E \cup EE \cup \cdots = \{s_1 s_2 \cdots s_k : k \in \mathbb{N}, s_k \in E\}.$$

- *We define the* star of a generating function F *to be*

$$F^* := 1 + F + F^2 + \cdots = \frac{1}{1-F}.$$

Now, let N be the negative walks in K, i.e., walks satisfying $y < x$, save for the first and last points. Let Φ be the nonpositive walks, i.e., walks in $y \le x$. Any negative walk is a right step followed by a nonpositive walk followed by an up step: $N = r\Phi u$. Note that *every* point of $w \in \Phi$ is counted as a losing time in rwu, so defining

$$\tilde{F}_\Phi := \sum_{w\in\Phi} z^{n(w)} t^{l(w)},$$

we have

$$F_N = zt^2 \tilde{F}_\Phi. \tag{2.1}$$

Next, any nonpositive walk consists of diagonal steps and negative walks. So $\Phi = d^*(Nd^*)^*$, and

$$\tilde{F}_\Phi = (zt)^*(F_N(zt)^*)^*. \tag{2.2}$$

It is child's play for Maple to solve (2.1) and (2.2) for $F_N(z,t)$. Now let P be the set of positive walks, i.e., walks in $y > x$, except for the endpoints. Positive walks are simply negative walks reflected about $y = x$, so $F_P = F_N(z, 1)$ (all positive walks have zero losing times).

Finally, any forward King walk consists of diagonal steps, negative walks, and positive walks: $K = d^*(Nd^* \cup Pd^*)^*$. In terms of generating functions,

$$F_K = z^*(F_N z^* + F_P z^*)^*,$$

and we are finished! We have $F_K(z,t)$ as an algebraic expression. Of course, it is rather messy, so we do not record it here. To see it for yourself, use `ForwardKingGF(z,t)` in the Maple package. Unfortunately, F_K is too complex to be amenable to either of the moment-finding methods discussed previously. However, we should not lose hope...

2.3 Numerically Analyzing Moment Asymptotics

2.3.1 Recursive Enumeration of Walks

We started with steps in $\{(1,0),(0,1)\}$. Then we added the diagonal step $(1,1)$. Now let us take the affair even further.

Definition 2.6. *Given $S \subset \mathbb{N}^2$, let W^S be the set of walks from $(0,0)$ with steps in S. For $(a,b) \in \mathbb{N}^2$, let $W^S_{a,b}$ contain walks of W^S ending at (a,b).*

In the $S = \{(1,0),(0,1)\}$ case, we were able to calculate the moments of $a_1|W^S_{n,n}$ in terms of n. We cannot expect to do this in general. Indeed, even in the (still very symmetric) case $S = \{(1,0),(0,1),(1,1)\}$, we could not find nice expressions for the moments.

However, we can *fix* $(a,b) \in \mathbb{N}^2$ and focus on the *finite* set of walks $W^S_{a,b}$. Then the generating function

$$F_{a,b}(t) := \sum_{w \in W^S_{a,b}} t^{a_1}$$

is a *finite* polynomial in t, with easily computable moments. Further, given fixed S and (a,b), we can make use of the fact that

$$W^S_{a,b} = \bigcup_{s \in S} W_{(a,b)-s}\{s\}$$

to compute $F_{a,b}(t)$ with a recursive procedure; this is done in `F2G`. So for each (a,b), the moments of $a_1|W^S_{a,b}$ are can be found with a computer.

2.3.2 Asymptotic Storybooks

The procedure `ChungFellerBook2D(S,M,K1,K2)` uses `F2G` to compute the expectation, variance, and central standardized moments three through M of $a_1|W^{S'}_{n,n}$ for $S' \subset S$, $n = K1, \ldots, K2$. It uses this data to guess the asymptotic behavior of the moments as functions of n. We use the ansatzes Cn for expectation, Cn^2, and C for the third and higher central standardized moments.

So, for each $S' \subset S$, a theorem about the asymptotic behavior of walks with steps in S' is generated (step sets producing trivial theorems are automatically excluded).

Of course, we must add the disclaimer that these "theorems" are merely numerical approximations to the asymptotic behavior of the moments. To be extra safe, the procedure `ChungFellerBook2DSafe` runs `ChungFellerBook2D` twice, with different n-ranges. For each theorem it computes the constants twice; then it only keeps the agreeing digits.

In the case $S = \{(1,0),(0,1)\}$, where we do know the moments as functions of n, we can confirm that `ChungFellerBook2DSafe` gives good results.

Table 2.1 summarizes the output of `ChungFellerBook2DSafe({[1,0],[0,1],[1,1],[2,0],[0,2]},6,100,110, 190,200);`. The zeroth column is the set of allowed steps, where for brevity $ij := (i,j)$. Columns 1-6 are the asymptotic expectation, variance, third through sixth standardized central moments. Note that by Proposition 2.2, the exact values of the first row are $n, n^2/3, 0, 1.8, 0, 27/7 \approx .38571$.

Steps	1	2	3	4	5	6
$\{01,10\}$	$1.0000n$	$0.3n^2$	0.0000	1.800	0.0000	3.86
$\{01,20\}$	$0.38n$	$0.1n^2$	0.0	0.900	-0.1	1.93
$\{02,20\}$	$0.2500n$	$0.043n^2$	0.0000	0.900	0.0000	1.93
$\{01,02,10\}$	$0.9n$	$0.27n^2$	0.0	1.8023	-0.03	3.87
$\{01,02,20\}$	$0.33n$	$0.07n^2$	-0.02	0.90	-0.1	2.
$\{01,10,11\}$	$0.8n$	$0.2n^2$	0.0	1.8	0.	3.9
$\{01,11,20\}$	$0.666n$	$0.15n^2$	0.001	1.80	0.01	3.9
$\{02,11,20\}$	$0.5n$	$0.08n^2$	0.0	1.80	0.	3.9
$\{01,02,10,11\}$	$0.81n$	$0.22n^2$	0.	1.80	0.0	3.9
$\{01,02,10,20\}$	$0.80n$	$0.21n^2$	-0.01	1.804	0.	4.
$\{01,02,11,20\}$	$0.6n$	$0.1n^2$	-0.01	1.80	-0.1	3.9
$\{01,10,11,20\}$	$0.81n$	$0.22n^2$	0.	1.80	0.03	3.89
$\{01,02,10,11,20\}$	$0.75n$	$0.19n^2$	-0.004	1.8	-0.011	4.

Table 2.1: Asymptotic moments for walks from $(0,0)$ to (n,n) for various step sets.

2.3.3 Walks in Higher Dimensions

This method easily generalizes to three or more dimensions. If we consider walks in $\mathbb{N}^3$, then we have seven statistics to keep track of the number of times the walk visits the regions $x < y < z$, $x < z < y$, $y < x < z$, $y < z < x$, $z < x < y$, $z < y < x$, and "none of the above." The corresponding generating function (over walks to a fixed point in $\mathbb{N}^3$) is computed in `F3G`. Not surprisingly, this procedure is significantly slower than `F2G`.

2.4 Conclusion

Many areas still need to be explored. For example, we have focused mainly on the number of losing times, a_1. But the method of Section 2.3 could also be applied to the other statistics in Definition 2.1. Also, there is much to be done with walks in higher dimensions. We encourage you to experiment with the Maple package and make discoveries of your own!

Chapter 3

Simultaneous Core Partitions

In the previous chapter, we examined an indexed collection of combinatorial objects (walks), and we used generatingfunctionology and dynamic programming to analyze the distribution of these objects, including the asymptotics. This general methodology can be applied to other objects; here, we will use it to study certain classes of integer partitions.

3.1 Introduction to Core Partitions

A *partition* of a positive integer n is a nonincreasing list of positive integers $(\lambda_1, \ldots, \lambda_k)$ summing to n. We say that n is the *size* of the partition, and $\lambda_1, \ldots, \lambda_k$ are the *parts*. The *Young diagram* is a way to graphically represent a partition as an arrangement of left-justified boxes, where there are λ_k boxes in the kth row from the top; see Figure 1.1.

The *hook length* of a box in the Young diagram of a partition is the number of boxes to the right (the arm) plus the number of boxes below it (the leg) plus one (the head). In Figure 1.1, the boxes are labeled with their hook lengths. A partition is an *s-core* if its Young diagram avoids hook length s and an *(s,t)-core* if it avoids hook lengths s and t [3]. Here, we will focus on these latter *simultaneous core* partitions. (Note: some literature defines a t-core to be a partition with no hook lengths *divisible* by t. We will *not* be using this definition.)

The number of (s,t)-core partitions is finite iff s and t are coprime, which we will assume from now on [3]. Let $X_{s,t}$ be the random variable "size of an (s,t)-core partition," where the sample space is the set of all (s,t)-core partitions, equipped with the uniform distribution. In [10], with the help of Maple, Zeilberger derived explicit expressions for the expectation, variance, and numerous higher moments of $X_{s,t}$. The original paper noted that "From the 'religious-fanatical' viewpoint of the current 'mainstream' mathematician, they are 'just' conjectures, but nevertheless, they are **absolutely certain** (well, at least as absolutely certain as most proved theorems)," and a donation to the OEIS was offered for the theory to make the results rigorous. Later, it was found that such theory did exist and the results are entirely rigorous; see the updates at the paper's site.

Zeilberger also computed some standardized central moments of $X_{s,t}$ and the limit of these expressions as $s, t \to \infty$ with $s - t$ fixed. From this he conjectured the limiting distribution. Perhaps surprisingly, it is abnormal.

Here, we will investigate what happens if we impose various additional restrictions on the simultaneous core partitions. We will start out by requiring that the parts be *distinct*. The analysis becomes harder in this case, and we will need to examine various sub-classes of partitions separately. Finally, we will give some results about simultaneous core partitions with *odd* parts.

3.2 Partitions with Distinct Parts that are $(s, s+1)$-Cores

This section is adapted from our paper [31].

For the case of (s, t)-cores with distinct parts, we do not know closed-formed expressions for the moments of the size in terms of s and t. Instead, we further restrict to certain indexed families with only one variable index. First, we will consider the simplest case: $(s, s+1)$-core partitions with distinct parts.

Amdeberhan [1] initiated the study of simultaneous core partitions with distinct parts and conjectured that the number of $(s, s+1)$-core partitions with distinct parts is given by the Fibonacci number F_{s+1}. This was proved by Armin Straub [22] and Huan Xion [26]. Xion also proved a conjectured expression of Amdeberhan for the expected size, in terms of a double sum involving Fibonacci numbers. He, we will go even further, finding the higher moments in terms of s and analyzing the limiting distribution as $s \to \infty$.

3.2.1 Computing the Generating Function

Given a positive integer s, let P_s be the set of all $(s, s+1)$-core partitions with distinct parts. Observe that $|P_s|$ is always finite, since $\gcd(s, s+1) = 1$. Let X_s be the random variable "size of a uniformly randomly chosen partition in P_s." Our goal is to have an efficient way to compute the generating function

$$G_s(q) := \sum_{p \in P_s} q^{|p|}$$

for fixed s. (Here $|p|$ denotes the size of a partition p.) This will then allow us to compute moments of X_s.

Recall that the *perimeter* of a partition is the size of the largest hook length. Straub [22] gives a useful characterization of P_s in terms of perimeters:

Lemma 3.1 (Lemma 2.2 of [22]). *A partition into distinct parts is an $(s, s+1)$-core iff it has perimeter $< s$.*

From this, Straub also proved Amdeberhan's [1] conjecture that the number of $(s, s+1)$-core partitions with distinct parts is given by the Fibonacci number:

$$|P_s| = G_s(1) = F_{s+1}. \tag{3.1}$$

Lemma 3.1 gives us a fast way to compute $G_s(q)$. Define $P_{k,l}$ to be the set of partitions with l distinct parts and largest part k. By the Lemma, a partition p is an $(s, s+1)$-core iff $p \in P_{k,l}$ for some $k + l \leq s$. Define

$$G_{k,l}(q) := \sum_{p \in P_{k,l}} q^{|p|}.$$

This generating function is computed recursively by `Gkl(q,k,l)` in the Maple package (keyword `Feller` in Appendix A). Finally, summing $G_{k,l}(q)$ for $k+l \leq s$ gives us $G_s(q)$, implemented in the procedure `Gs(q,s)` in the Maple package.

3.2.2 Conjecturing the Moments

Using `Gs(q,s)`, we can now compute moments of X_s for a given s. If we define the operator $L : f(q) \mapsto qf'(q)$, then the kth moment of X_s is

$$\mathbb{E}[X_s^k] = \left.\frac{L^k(G_s(q))}{G_s(q)}\right|_{q=1} = \frac{L^k(G_s(q))(1)}{F_{s+1}}, \tag{3.2}$$

where we have used (3.1).

Suppose we fix k. Then the numerator, call it $P(s)$, in (3.2) depends only on s. Experimental evidence indicates that $P(s)$ is of the form $A(s)F_s + B(s)F_{s+1}$ for some polynomials A, B. Further, we can use the procedure `GuessFibPol(L,n)` to guess A, B from computed values of $P(s)$.

To summarize, we conjecture that for k fixed, there exist polynomials $A(s), B(s)$ such that

$$\mathbb{E}[X_s^k] = A(s)\frac{F_s}{F_{s+1}} + B(s), \tag{3.3}$$

and (for fixed k), these polynomials can be guessed from data supplied by `Gs(q,s)`.

3.2.3 Proving the Conjectures

Now we go over the theory needed to validate the above conjectures. Recall that the q-binomial coefficient $\binom{m+n}{m}_q$ gives the generating function for partitions whose Young diagrams fit inside an $m \times n$ rectangle. In other words, $\binom{m+n}{m}_q$ is the sum of $q^{|p|}$, where p ranges over partitions with $\leq m$ parts and largest part $\leq n$. Let us denote these by "$m \times n$ partitions."

Lemma 3.2. *The generating function (according to size) of partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ with m distinct parts, each satisfying $\lambda_i \leq n$, is*

$$\sum_{k \leq n} G_{k,m}(q) = q^{\binom{m+1}{2}} \binom{n}{m}_q.$$

Thus,

$$G_s(q) = \sum_{m=0}^{s} \sum_{k \leq s-m} G_{k,m}(q) = \sum_{m=0}^{s} q^{\binom{m+1}{2}} \binom{s-m}{m}_q.$$

Proof. Note that $\binom{n}{m}_q$ is the generating function of $m \times (n-m)$ partitions. Given such a partition p, we can add $1, 2, 3, \ldots, m$ to its parts (counting missing parts as having size 0), producing a partition with exactly m distinct parts of size $\leq n$. This increases $|p|$ by $\binom{m+1}{2}$. Further, it is easy to see that this operation defines a bijection. $\square$

Now, since $G_s(q)$ is expressed as a q-binomial sum, the theory developed by Wilf and Zeilberger in [24] guarantees that $G_s(q)$ satisfies a recurrence. We use the procedure `qGuessRec` in our Maple package to guess the recursion from the first, say, 30 terms of the sequence $\{G_s(q)\}_s$, obtaining the following:

$$\begin{aligned}
G_1(q) &= 1 \\
G_2(q) &= 1 + q \\
G_3(q) &= q^2 + q + 1 \\
G_4(q) &= 2q^3 + q^2 + q + 1 \\
G_s(q) &= G_{s-1}(q) + q^{s-1}G_{s-3}(q) + q^{s-1}G_{s-4}(q).
\end{aligned} \tag{3.4}$$

Later, in hindsight, we were able to derive this recurrence straight from the formula for $G_s(q)$ in Lemma 3.2. We used Zeilberger's Maple package qEKHAD (see the book [19]), which is capable of both finding and rigorously proving recurrences satisfied by q-binomial sums such as the one in our Lemma.

From (3.4), it follows that the *moments* of the sequence $\{G_s(q)\}_s$ obey the C-finite ansatz. That is, they satisfy linear recurrences with constant coefficients; see [40]. Thus, we need only check our conjectures for finitely many values of s to prove them. (In practice, we checked for 70 values of s to compute expressions for up to the sixteenth moment.)

With these observations and the help of Maple, we are now ready to find explicit expressions for the moments of X_s. Fix k. We use the recursion (3.4) to efficiently compute the k^{th} moment of X_s for many values of s. Following Section 3.2.2, we then conjecture an expression for the k^{th} moment of X_s which fits the template from (3.3). By the above argument, our conjectured expression is proven for all s if it holds for sufficiently many values of s.

For moments two and higher, it is more meaningful to compute the *central moment.* Recall that the k^{th} central moment of X is $\mathbb{E}[(X-\mu)^k]$, where μ is the expectation. For example, the second central moment is the variance.

Expressions for up to moment 16 may be found in the Maple output file `theorems.txt` in Appendix A. Here is a small sample of the results:

Theorem 3.3. *Let X_s be the random variable "size of a uniformly random $(s, s+1)$-core partition with distinct parts," and F_s denote the sth Fibonacci number. Then,*

(i)

$$\mathbb{E}[X_s] = \frac{1}{50}\,\frac{5\,s^2F_{s+1} - 6\,sF_s + 7\,sF_{s+1} - 6\,F_s}{F_{s+1}}.$$

(ii)

$$\begin{aligned} Var(X_s) = {} & \\ & (20\,s^3F_sF_{s+1} + 10\,s^3{F_{s+1}}^2 - 27\,s^2{F_s}^2 + 33\,s^2F_sF_{s+1} \\ & + 57\,s^2{F_{s+1}}^2 - 54\,s{F_s}^2 - 32\,sF_sF_{s+1} + 65\,s{F_{s+1}}^2 \\ & - 27\,{F_s}^2 - 45\,F_sF_{s+1})/(1875F_{s+1}^2). \end{aligned}$$

(iii) The third central moment of X_s is asymptotic to

$$\begin{aligned} & -(3/31250)(65s^4\phi^3 - 40s^4\phi^2 + 222s^3\phi^3 - 40s^4\phi - 218s^3\phi^2 \\ & - 65s^2\phi^3 - 106s^3\phi - 338s^2\phi^2 - 390s\phi^3 + 36s^3 - 2s^2\phi + 110s\phi^2 \\ & + 108s^2 + 154s\phi + 270\phi^2 + 108s + 90\phi + 36)\phi^{-3}, \end{aligned}$$

where ϕ is the Golden Ratio.

Note that in (iii), we print the asymptotic result simply because the exact expression would take up too much space. Also, (i) is an explicit version of Conjecture 11.9(d) made by Amdeberhan [1] and later proven by Xiong [26].

3.2.4 Limiting Distribution

Recall that the kth standardized central moment of X is $\mathbb{E}[(X-\mu)^k]/\sigma^k$, where μ is the expectation and σ is the standard deviation, and the normal distribution has a sequence of standardized central moments which alternates between 0 and odd factorials:

$0, 1, 0, 3, 0, 15, 0, 105, 0, 945, \ldots$.

In `theorems.txt`, the limit as $s \to \infty$ of the first 16 standardized central moments of X_s are shown to coincide with that of the normal distribution, giving strong evidence for the following:

Conjecture 3.4. *X_s is asymptotically normal. That is, the distribution of $(X_s - \mathbb{E}[X_s])/\sqrt{Var(X_s)}$ converges to the standard normal distribution as $s \to \infty$.*

Note that in [10], the limiting distribution of "size of an (s,t)-core partition" (with the distinct parts condition dropped) was proven to follow an *ab*normal distribution.

An approach inspired by [38] might be useful in proving Conjecture 3.4. The main idea is to keep track of the *leading terms* in the expressions of the moments, and perhaps use (3.4) to derive a recurrence for the limiting moments.

3.3 Partitions with Distinct Parts that are $(2s+1, 2s+3)$-cores

This section is adapted from our paper [32].

At the end of his beautiful paper, [22] (where, among many things, the author describes a beautiful new elegant partition identity between odd and distinct integer partitions which preserves the perimeter), Armin Straub conjectured two intriguing enumeration results:

Theorem 3.5 (conjectured in [22], first proved in [28])**.** *The number of $(2s+1, 2s+3)$-core partitions with distinct parts equals 4^s.*

Theorem 3.6 (conjectured in [22], first proved in [28])**.** *The largest size of a $(2s+1, 2s+3)$-core partition with distinct parts is $\frac{1}{24}\,(5\,s+11)\,s\,(s+2)\,(s+1)$.*

The proofs in [28] use ingenious, but rather complicated, combinatorial arguments. Here, we will give new, much simpler, "experimental-mathematical" proofs, that can be easily made rigorous. But our main purpose is to establish *explicit* expressions for the expectation, variance, and all the moments up to the seventh. With more computing power, it should be possible to go beyond. We then go on and use these explicit (polynomial) expressions in order to find the limits of the scaled moments, giving exact values for the first seven moments of the limiting (scaled) probability distribution of the random variable "size" over $(2s+1, 2s+3)$-core partitions with distinct parts (as $s \to \infty$). Professor Zeilberger has promised to donate \$100 to the OEIS foundation for identifying that limiting (continuous) probability distribution.

3.3.1 Explicit Expressions for the First Seven Moments

First, we will summarize the key results.

Theorem 3.7. *The average size of a* $(2s+1, 2s+3)$*-core partition with distinct parts is*

$$\frac{1}{32}(10\,s^3 + 27\,s^2 + 19\,s).$$

Note that the corresponding average taken over *all* partitions, according to Armstrong's ex-conjecture, is $\frac{1}{6}s(s+1)(2s+5) = \frac{1}{3}s^3 + O(s^2)$, while, according to Theorem 3.7, our average (i.e. for the distinct case) is $\frac{5}{16}s^3 + O(s^2)$, so it is a bit less.

Theorem 3.8. *The variance of the random variable "size" defined on the set of* $(2s+1, 2s+3)$*-core partitions with distinct parts is*

$$\frac{1}{15360}(934\,s^6 + 4687\,s^5 + 9700\,s^4 + 10505\,s^3 + 6256\,s^2 + 1518\,s).$$

Note that according to [10], the corresponding variance, taken over *all* partitions is

$$\frac{1}{720}\,(2\,s+1)\,(2\,s+3)\,(2\,s+2)\,s\,(4\,s+5)\,(4\,s+4)$$

which is $\frac{8}{45}s^6 + O(s^5) = 0.1777777778 s^6 + O(s^5)$, while for our case, according to Theorem 2, it is $\frac{467}{7680}s^6 + O(s^5) = 0.06080729167 s^6 + O(s^5)$.

Theorem 3.9. *The third moment (about the mean) of the random variable "size" defined on* $(2s+1, 2s+3)$*-core partitions with distinct parts is*

$$\frac{1}{27525120} \cdot (793586\,s^9 + 4945025\,s^8 + 12775144\,s^7 + 17215282\,s^6 + 11839450\,s^5$$
$$+1535905\,s^4 - 4756804\,s^3 - 4342612\,s^2 - 1297776\,s).$$

Theorem 3.10. *The fourth moment (about the mean) of the random variable "size" defined on $(2s+1, 2s+3)$-core partitions with distinct parts is*

$$\frac{1}{54499737600} \cdot (1743712560\, s^{12} + 13490284234\, s^{11} + 45408125279\, s^{10}$$
$$+87568584895\, s^{9} + 109173019890\, s^{8} + 97494786972\, s^{7} + 68082466947\, s^{6}$$
$$+34594762895\, s^{5} + 8734303600\, s^{4} + 3269131844\, s^{3} + 7648567524\, s^{2} + 4135638960\, s).$$

Theorem 3.11. *The fifth moment (about the mean) of the random variable "size" defined on $(2s+1, 2s+3)$-core partitions with distinct parts is*

$$\frac{1}{108825076039680} \cdot s\,(s+1)\,(4115597238066\, s^{13} + 30331407775461\, s^{12}$$
$$+93240357590320\, s^{11} + 153901186416765\, s^{10} + 154511084293844\, s^{9}$$
$$+126787455814599\, s^{8} + 115227024155664\, s^{7} + 42586120680111\, s^{6}$$
$$-95604599727502\, s^{5} - 105409116317640\, s^{4} + 43165327777096\, s^{3}$$
$$+91113907956144\, s^{2} - 30975685518528\, s - 65049004454400).$$

Theorem 3.12. *The sixth moment (about the mean) of the random variable 'size' defined on $(2s+1, 2s+3)$-core partitions with distinct parts is*

$$\frac{1}{8288117791182028800} \cdot$$
$$(459077029253573970\, s^{18} + 3986958940758529155\, s^{17} + 14588638597341766281\, s^{16}$$
$$+29315654117562943844\, s^{15} + 38855616058049391120\, s^{14} + 52048632801161949890\, s^{13}$$
$$+87053992212835094382\, s^{12} + 102228197171521441748\, s^{11} + 24538654588404043230\, s^{10}$$
$$-81063397918244586845\, s^{9} - 37681424022539337807\, s^{8} + 128753068232342353072\, s^{7}$$
$$+136357236921377110920\, s^{6} - 109095423240535042640\, s^{5} - 264555566724556223856\, s^{4}$$
$$-62480060539123323264\, s^{3} + 164786511770490504960\, s^{2} + 100625844884387235840\, s)\,.$$

Theorem 3.13. *The seventh moment (about the mean) of the random variable 'size' defined on* $(2s+1, 2s+3)$*-core partitions with distinct parts is*

$$\frac{s(s+1)}{2^{40}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19}\cdot$$

$$(20325334435585878 4830\, s^{19} + 1525941518277673062635\, s^{18}$$

$$+4376090780890032310694\, s^{17} + 5920532244827036954724\, s^{16}$$

$$+7108181147332994381598\, s^{15} + 2251661486261904165744 0\, s^{14}$$

$$+4773775443254246875071 0\, s^{13} + 2143153818338605219130 6\, s^{12}$$

$$-7712734979094522122165 2\, s^{11} - 9878860853094467978210 7\, s^{10}$$

$$+9146862817518869990074 8\, s^{9} + 2761985949218219059930 26\, s^{8}$$

$$+1643105926798936520735 04\, s^{4} + 1420837514400804031281 984\, s^{3}$$

$$+5315267935858391947536 0\, s^{7} - 5163746794374759608700 16\, s^{6}$$

$$-6969412242969426556873 12\, s^{5} + 1109985197630308975715 328\, s^{2}$$

$$-7459510615037154546739 20\, s - 1026387551269849288826 880).$$

Here are some corollaries of the theorems above:

1. The limit of the "coefficient of variation" (the quotient of the standard deviation to the mean), as $s \to \infty$, is $\frac{1}{150}\sqrt{14010} = 0.78909230554268279 89\ldots$. In particular, since that limit is *not* zero, unlike $(k, k+1)$-core partitions with distinct parts discussed in [31], there is *no* concentration about the mean.

2. The limit of the *skewness*, as $s \to \infty$, is $\frac{396793}{390815488}\sqrt{467}\sqrt{7680}$ $= 1.922787480888358667\ldots$

3. The limit of the *kurtosis*, as $s \to \infty$, is $\frac{145309380}{16792853} = 8.6530490084085\ldots$

4. The limit of the scaled fifth moment (α_5), as $s \to \infty$, is $\frac{3429664365055}{156594294624768}\sqrt{467}\sqrt{7680} = 41.4777067204457\ldots$

5. The limit of the scaled sixth moment (α_6), as $s \to \infty$, is $\frac{382564191044644975}{1552893421695616} = 246.35572905\ldots$.

6. The limit of the scaled seventh moment (α_7), as $s \to \infty$, is $\frac{56459262321071884675}{62988906654652346368}\sqrt{467}\sqrt{7680} = 697.5015509357\ldots$

3.3.2 Proving the Theorems

We now explain the methods used to obtain the results in the previous subsection.

The way Jaclyn Anderson proved her celebrated theorem [2] that if $\gcd(s,t) = 1$, then the number of (s,t)-core partitions equals $(s+t-1)!/(s!t!)$ was by defining a bijection with the set of *order ideals* of the poset

$$P_{s,t} := \mathbb{N} \backslash (s\mathbb{N} + t\mathbb{N}),$$

where $\mathbb{N} = \{0, 1, 2, 3, \ldots, \}$ is the set of non-negative integers, and the partial-order relation $c \leq_P d$ holds whenever $d - c$ can be expressed as $\alpha s + \beta t$ for some $\alpha, \beta \in \mathbb{N}$.

The set of order ideals of $P_{s,t}$, in turn, is in bijection with the set of *lattice paths* in the two-dimensional square lattice, from $(0,0)$ to (s,t) lying above the line $sy - tx = 0$. However, for our present purposes it is more efficient to use order ideals.

Recall that an order ideal I, in a poset P, is a set of vertices of P such that if $c \in I$ then all elements, d, such that $d \leq_P c$ also belong to I. Equivalently, if d does *not* belong to I, then all vertices c "above" it (i.e. such that $c \geq_P d$), also do *not* belong to I.

Let $n(s)$ be the number of order ideals of the lattice $P_{2s+1,2s+3}$ with no consecutive labels. Recall that, thanks to Jaclyn Anderson, this is the number of $(2s+1, 2s+3)$-core partitions with distinct parts, our *object of desire*.

Let's try and find an algorithm to compute the sequence $\{n(s)\}$ for as many terms as possible.

Let's review first how to prove that the number of order ideals of $P_{k+1,k+2}$, let's call it $p(k)$, is the Catalan number C_{k+1}. A plot of $P_{k+1,k+2}$ for $k = 8$ is shown in Figure 3.1. Note that the point $(k-1, 0)$ is labeled 1, and when we read the labels along diagonals, from the bottom-right to the top-left, the labels increase by 1, but as we move from the end of one diagonal to the next one there are "discontinuities" of sizes $3, 4, \ldots, k+1$ respectively. Let i be the smallest empty label on the hypotenuse, implying that $1, \ldots, i-1$ are occupied. We have to "kick out" all vertices that are $\geq_P$ of the vertex labeled i, leaving us with two connected components, triangles of sizes $i-2$ and $k-i$, with independent decisions regarding their order ideals. The "initial conditions" are $p(-1) = 1$, $p(0) = 1$, and for $k \geq 1$, we have

$$p(k) = \sum_{i=1}^{k+1} p(i-2)p(k-i). \tag{3.5}$$

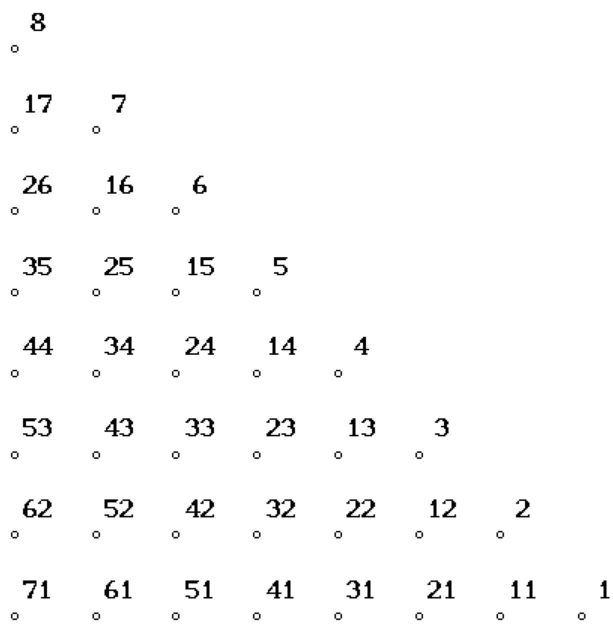

Figure 3.1: The lattice $P_{9,10}$.

Now let us move on to finding $n(s)$, i.e. the number of order ideals of $P_{2s+1,2s+3}$ without consecutive labels.

A diagram of the lattice $P_{2s+1,2s+3}$ (for $s = 6$) can be found in Figure 3.2 (see also Figure 3 in [28], where the lattice is drawn such that the rank-zero vertices are at the bottom rather than on the diagonal).

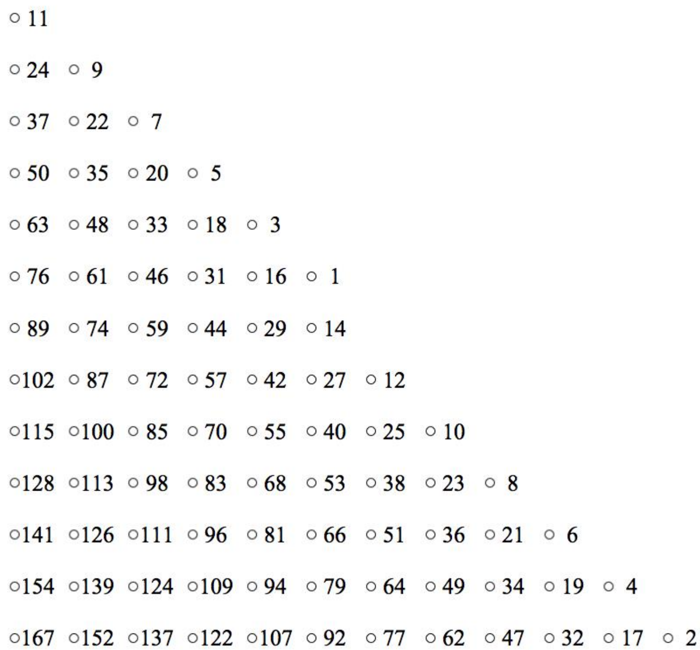

Figure 3.2: The lattice $P_{13,15}$.

Inspired by the reasoning in [28], let $2i-1$ $(1 \leq i \leq k)$, be the smallest odd vertex (of rank 0) that is *unoccupied*. This means that the vertices labeled $1, 3, \ldots, 2i-3$ are *occupied*. This means that the vertices with even labels, $2, \ldots, 2i-2$ are *unoccupied*, and since we are talking about *order ideals*, everything $\geq$ the odd vertex $2i-1$ and above the even vertices $2, \ldots, 2i-2$ gets kicked out, and for this scenario, we are left with counting order ideals of a smaller lattice, with two connected components, that consists of an even-labeled component, a triangle-lattice whose rank zero level has size s, and whose labels are $2i, 2i+2, \ldots, 2i+2s-2$, and an odd-labeled component, a triangle whose rank zero level has $s-i$ vertices, and whose labels are $2i+1, 2i+3, \ldots, 2s-1$. In addition, we have the definitely occupied vertices $1, \ldots, 2i-3$, but since they are definitely occupied, they don't contribute anything to the count of order ideals.

Figure 3.3 depicts the case when labels 1 and 3 of $P_{13,15}$ are occupied and 5 is empty. All vertices $\geq 5, 2, 4$ cannot be part of the order ideal.

Let $EO(a, b)$ be a two-triangle lattice, consisting of a triangle with a rank-zero vertices whose labels are $2, \ldots, 2a$, and a triangle of length-side b $(b > a)$ whose labels are $1, 3, \ldots, 2b-1$. (See Figure 3.4 for a picture of $EO(7, 9)$.) Going back to the

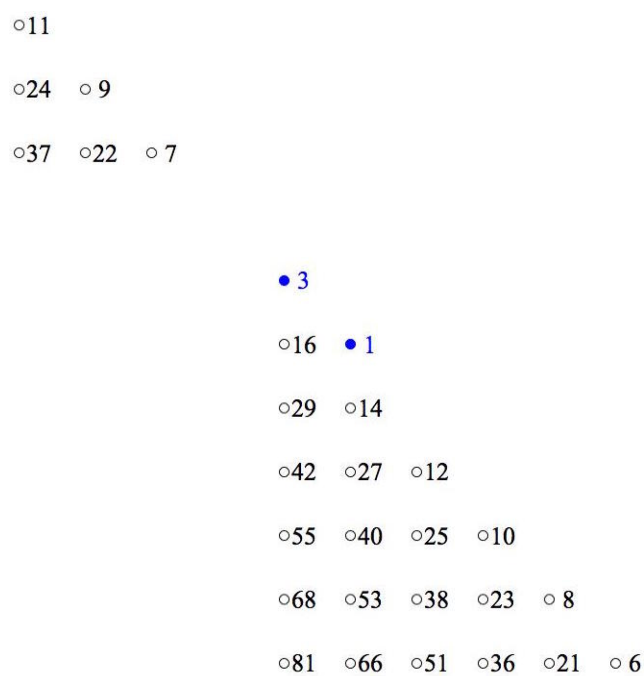

Figure 3.3: An order ideal of $P_{13,15}$ with smallest unoccupied odd label 5 and no consecutive labels must be a subset of the lattice pictured.

paragraph above, subtracting $2i-1$ from all labels gives us a lattice isomorphic to $EO(s-i,s)$. Let $e(a,b)$ be the number of order ideals of the lattice $EO(a,b)$ without consecutive labels. Then we have

$$n(s) = \sum_{i=1}^{s+1} e(s-i,s). \tag{3.6}$$

So if we could have an efficient scheme to compute $e(a,b)$, then we would be able to compute our sequence of desire, $n(s)$.

For $a \le b$, let $OE(a,b)$ be $EO(b,a)$, and let $o(a,b)$ be the number of order ideals without consecutive labels of $OE(a,b)$.

By looking at the smallest unoccupied odd-labeled vertex, say $2i-1$ (see Figure 3.5) we get, for $a \ge 1$:

$$e(a,b) = \sum_{i=1}^{b+1} o(a+1-i,b-i)\,p(i-2), \tag{3.7}$$

Figure 3.4: The lattice $EO(7, 9)$.

and for $a \leq 0$, we have $e(a, b) = p(b)$. Similarly, for $a \geq 1$,

$$o(a, b) = \sum_{i=1}^{a+1} e(a - i, b + 1 - i)\, p(i - 2), \tag{3.8}$$

and for $a \leq 0$, we have $o(a, b) = p(b)$.

The scheme consisting of equations (3.5)-(3.8) enables a very fast computation of the sequence $n(i)$, for, say $i \leq 400$, confirming, empirically for now, that $n(i) = 4^i$. However, this can be easily turned into a fully rigorous proof. A *holonomic description* (see [35], beautifully implemented by Christoph Koutschan in [17]) of both $e(a, b)$ and $o(a, b)$ can be readily guessed, and then, along with $p(k) = C_{k+1}$, the resulting identities (3.6)-(3.8) are routinely verifiable identities in the holonomic ansatz that can be plugged into Koutschan's "holonomic calculator." But since we know *a priori* that $n(k)$ satisfies *some* such recurrence, and it is extremely unlikely that its order is very high, confirming it for the first 400 values gives a convincing *semi-rigorous proof* that is easily rigorizable, if desired.

Note: In an e-mail correspondence, Armin Straub gave a far slicker, less computer-heavy, way, to conclude this experimental mathematics proof. See [23].

So much for enumeration of these partitions. Next, our goal is to obtain $(2s + 1, 2s + 3)$-analogs of our results in Section 3.2. Namely, we want to get data for the

Figure 3.5: A sub-lattice of $EO(7,9)$ which contains all order ideals of $EO(7,9)$ with smallest unoccupied odd label 9 and no consecutive labels.

expectation, variance, and moments of the size of the partitions in question. Thus, we need an efficient way to generate as many terms of the sequence of *Straub polynomials*, $S_s(q)$, defined by

$$S_s(q) := \sum_{p} q^{\text{size}(p)},$$

where p ranges over all $(2s+1, 2s+3)$-core partitions with distinct parts, and size(p) is the sum of the entries of p (i.e. the number of boxes in its Young Diagram).

The following method, which easily produced the first 21 Straub polynomials, is a weighted analog of the above order ideal-based enumeration scheme.

For an order ideal of $P_{s,t}$ let its *weight* be

$$q^{\text{Sum of Labels}} t^{\text{Number of Vertices}}.$$

Let $Q(s)$ be the set of order ideals of $P_{2s+1,2s+3}$ without neighboring labels (i.e. if $a \in I$ then both $a-1$ and $a+1$ are not in I). We define the two-variable polynomials

$$A_s(q,t) := \sum_{I \in Q(s)} q^{\text{Sum of Labels}(I)} t^{\text{Number of Vertices}(I)}.$$

Define the *umbra* (linear functional on polynomials of t) by

$$U(t^k) := q^{-k(k-1)/2},$$

and extended linearly. As shown by Anderson, once $A_s(q,t)$ are known, we get $S_s(q)$ by the transformation

$$S_s(q) = U(A_s(q,t)).$$

In other words, replace any power t^k that appears in $A_s(q,t)$ by $q^{-k(k-1)/2}$.

It remains to find an efficient scheme for cranking out as many terms of $A_n(q,t)$ that our computer would be willing to compute.

We first need a weighted analog of Equation (3.5), i.e., the weight-enumerator of $P_{k+1,k+2}$, but we need the extra generality where (still with the smallest label being 1), for *any* positive integers c and h, in the vertical direction it is going down by c, and in the horizontal direction it going down by $c+h$ (drawing the lattice so that the highest label, $1+(c+h)(k-1)$ is at the origin, and the vertex labeled 1 is situated at the point $(k-1,0)$, and the vertex labeled $1+(k-1)h$ is situated at the point $(0,k-1)$). Note that the original $P_{k+1,k+2}$ corresponds to $c=k+1$ and $h=1$.

Let's call this generalized weight-enumerator $P_k^{(c,h)}(q,t)$. It is readily seen that the weighted analog of (3.5) is

$$P_k^{(c,h)}(q,t) \;=\; \sum_{i=1}^{k+1} t^{i-1} \cdot q^{(i-1)+(i-1)(i-2)h/2} \cdot P_{i-2}^{(c,h)}(q,q^{c+h}t) \cdot P_{k-i}^{(c,h)}(q,q^{ih}t), \tag{3.9}$$

with the initial conditions $P_{-1}=1, P_0=1$.

Let $E_{x,y}^{(c)}(q,t)$ be the weight-enumerator of the lattice $EO(x,y)$ with horizontal spacing c and vertical spacing $c+2$. Then the weighted analog of (3.6) is

$$A_s(q,t) \;=\; \sum_{i=1}^{s+1} t^{i-1} q^{(i-1)^2} \cdot E_{s-i,s}^{(2s+1)}(q,q^{2i-1}t). \tag{3.10}$$

Let $O_{x,y}^{(c)}(q,t)$ be the weight-enumerator of the lattice $OE(x,y)$, with horizontal spacing c and vertical spacing $c+2$. Then the weighted analog of (3.7) can be seen to be

$$E_{x,y}^{(c)}(q,t) \;=\; \sum_{i=1}^{y+1} t^{i-1} \cdot q^{(i-1)^2} \cdot O_{x-i+1,y-i}^{(c)}(q,q^{2i-1}t) \cdot P_{i-2}^{(c,2)}(q,q^{c+2}t), \tag{3.11}$$

with the initial condition $E_{x,y}^{(c)}(q,t) = P_y^{(c,2)}(q,t)$ when $x \le 0$.

Finally, the weighted analog of (3.8) is

$$O_{x,y}^{(c)}(q,t) \;=\; \sum_{i=1}^{x+1} t^{i-1}\, q^{(i-1)^2} \,\cdot\, E_{x-i,y-i+1}^{(c)}(q, q^{2i-1}t) \,\cdot\, P_{i-2}^{(c,2)}(q, q^{c+2}t), \tag{3.12}$$

with the initial condition $O_{x,y}^{(c)}(q,t) = P_y^{(c,2)}(q,qt)$ when $x \le 0$.

3.3.3 The first 21 Straub Polynomials

Using the above scheme, one gets that

$$S_1(q) \;=\; q^4 + q^2 + q + 1,$$

$$S_2(q) \;=\; q^{21} + q^{16} + 2\,q^{12} + q^9 + q^8 + q^7 + q^6 + q^5 + 2\,q^4 + 2\,q^3 + q^2 + q + 1$$

$$S_3(q) \;=\; q^{65}+q^{56}+q^{48}+q^{47}+q^{41}+q^{39}+q^{37}+2\,q^{35}+q^{32}+q^{30}+2\,q^{29}+q^{28}+q^{26}+3\,q^{24}+q^{23}$$
$$+q^{22}+q^{21}+q^{20}+2\,q^{19}+2\,q^{18}+3\,q^{17}+q^{16}+q^{15}+2\,q^{14}+2\,q^{13}+2\,q^{12}+3\,q^{11}+q^{10}+3\,q^9+3\,q^8$$
$$+3\,q^7 + 4\,q^6 + 3\,q^5 + 2\,q^4 + 2\,q^3 + q^2 + q + 1,$$

$$S_4(q) \;=\; q^{155}+q^{141}+q^{128}+q^{125}+q^{116}+q^{112}+2\,q^{105}+q^{103}+q^{100}+2\,q^{95}+q^{93}+q^{91}+2\,q^{89}+q^{85}$$
$$+q^{84}+q^{83}+2\,q^{82}+q^{80}+q^{79}+q^{78}+q^{76}+q^{74}+q^{73}+q^{72}+2\,q^{71}+2\,q^{70}+q^{69}+2\,q^{68}+q^{67}+q^{65}+q^{64}$$
$$+q^{63}+5\,q^{61}+q^{60}+2\,q^{59}+3\,q^{57}+q^{56}+3\,q^{55}+4\,q^{53}+2\,q^{52}+2\,q^{51}+2\,q^{50}+q^{49}+2\,q^{48}+3\,q^{47}$$
$$+2\,q^{46}+3\,q^{45}+4\,q^{44}+2\,q^{43}+q^{42}+5\,q^{40}+3\,q^{39}+4\,q^{38}+5\,q^{37}+2\,q^{36}+3\,q^{35}+q^{34}+4\,q^{33}$$
$$+6\,q^{32}+5\,q^{31}+3\,q^{30}+4\,q^{29}+3\,q^{28}+5\,q^{27}+4\,q^{26}+7\,q^{25}+5\,q^{24}+6\,q^{23}+3\,q^{22}+4\,q^{21}+5\,q^{20}$$
$$+5\,q^{19}+4\,q^{18}+5\,q^{17}+6\,q^{16}+5\,q^{15}+4\,q^{14}+7\,q^{13}+6\,q^{12}+7\,q^{11}+7\,q^{10}+6\,q^9+6\,q^8+5\,q^7$$
$$+4\,q^6 + 3\,q^5 + 2\,q^4 + 2\,q^3 + q^2 + q + 1.$$

For the Straub polynomials $S_s(q)$ for $5 \le s \le 21$, see the output file or use procedure `ASpc(s,q)` in the Maple package `Armin.txt` listed in Appendix A.)

Unlike the case of $(s, s+1)$-core partitions, whose number happened to be F_{s+1}, and the explicit expressions for the expectation, variance, and higher moments involved expressions in F_s, F_{s+1} and s, the present case of $(2s+1, 2s+3)$-core partitions into distinct parts, gives “nicer” results. This is because, as conjectured in [22] and first proved in [28] (and reproved above), the actual enumeration is as simple as can be, namely 4^s. Hence it is not surprising that the expectation, variance, and higher moments are *polynomials* in s.

To get expressions for the moments we used the empirical-yet-rigorizable approach of [38] and [39], as follows.

Using the first 21 Straub polynomials, we get the sequence of numerical averages $S_s'(1)/4^s$, $1 \leq s \leq 21$, and fit it to a polynomial of degree 3 (in fact four terms suffice!), we giving the expression for the expectation, let's call it $\mu(s)$, stated in Theorem 3.7 above.

Using the sequence

$$\frac{(q\frac{d}{dq})^2 S_s(q)|_{q=1}}{4^s} - \mu(s)^2,$$

for $1 \leq s \leq 7$, and fitting it with a polynomial of degree 6, we get an explicit expression for the variance, thereby getting Theorem 3.8. The conjectured polynomial expression agrees all the way to $s = 21$.

The third through seventh moments are derived similarly, where the ith moment (about the mean, but also the straight moment) turns out to be a polynomial of degree $3i$ in s.

Let us comment that we strongly believe that all the results here can be, *a posteriori*, justified rigorously. The complicated functional recurrences for the Straub polynomials (before the "umbral application") entail, after Taylor expansions about $q = 1$, extremely complicated recurrence relations for the (pre-) moments, whose details do not concern us, since we know that their truth follows by induction. The reason that we are not completely sure about this is that we don not have a formal proof that "polynomiality" is preserved under the umbral transform. Granting this, each such identity is a *polynomial identity*, and hence its truth follows from plugging in sufficiently many special cases. But that is how we got them in the first place. *QED!*

3.3.4 A Short Proof of Straub's Ex-Conjecture About the Maximal Size

In [28], the authors used quite a bit of *human ingenuity* to prove Armin Straub's conjecture (posed in [22]) that the maximal size of a $(2s+1, 2s+3)$-core partition into distinct parts is given by the degree 4 polynomial $\frac{1}{24}\,(5\,s+11)\,n\,(s+2)\,(s+1)$.

We strongly believe that one can deduce from general, *a priori*, *hand-waving* (yet fully rigorous) considerations that this quantity is *some* polynomial of degree ≤ 5. Hence it is enough to check it for $1 \leq s \leq 6$. But this quantity is exactly the *degree* of the Straub polynomial $S_s(q)$. We verified it, in fact, all the way to $s = 21$, so Theorem 3.6 is re-proved (modulo our belief).

3.4 Partitions with Distinct Parts that are $(s, ds-1)$-Cores

This section is adapted from our paper [30].

In [22], Straub generalizes the problem in Section 3.2 by considering $(s, ds-1)$-core partitions with distinct parts, where s and d are natural numbers. He proves in Theorem 4.1 that the number of such partitions, call it $N_d(s)$, satisfies a generalized Fibonacci recurrence:

$$\begin{aligned} &N_d(1) = 1,\ \ N_d(2) = d, \\ &N_d(s) = N_d(s-1) + dN_d(s-2). \end{aligned} \tag{3.13}$$

Of course, this reduces to the usual Fibonacci numbers when $d = 1$. Note that we can view $N_d(s)$ as a sequence of polynomials in d.

Here, we will use the poset characterization in Section 3.3.2 to easily recover Straub's result and discover new conjectures about the distribution of the sizes of the partitions.

3.4.1 Understanding the Posets

By Section 3.3.2, we know that $(s, ds-1)$-core partitions with distinct parts are bijective with order ideals of $P_{s,ds-1}$ containing no consecutive labels. We can use the procedure `PW` in the Maple package `Armin` (see Appendix A) to plot $P_{s,ds-1}$ for various s and d.

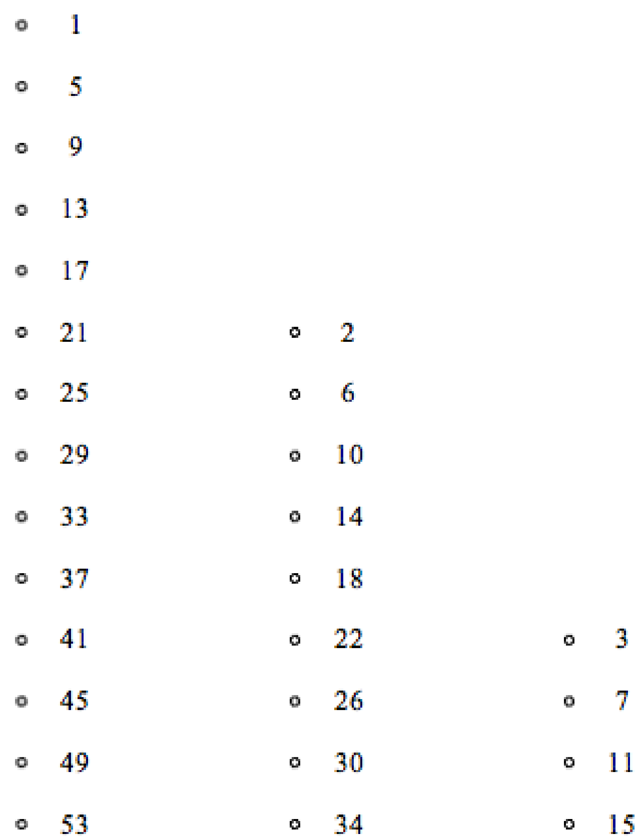

Figure 3.6: The poset $P_{4,19} = P_{4,4\cdot 5-1}$.

For example, Figure 3.6 depicts the poset $P_{4,19}$, i.e., the $s = 4, d = 5$ case. (When plotting $P_{s,t}$, we use the convention to increment s in the $\downarrow$ direction and t in the $\leftarrow$ direction. Thus, the largest label, $st - s - t$, is in the lower left corner.) It is easy to show that this general trapezoidal shape persists for arbitrary values of s and d. We can also see $P_{s,ds-1}$ as a colonnade of $s-1$ vertical pillars with heights $d(s-1)-1, d(s-2)-1, \ldots, d-1$. Further, the tops of the pillars have labels $1, 2, \ldots, s-1$.

3.4.2 Characterizing the Order Ideals

Next, we recover the recursion (3.13) by enumerating the order ideals of $P_{s,ds-1}$ with no consecutive labels.

Referring to the $s = 4, d = 5$ example, let I be an order ideal of $P_{4,19}$. Let I_k be the part of I contained in the kth pillar.

- If $I_1 = \emptyset$, then I is isomorphic to an order ideal of $P_{s-1,d(s-1)-1} = P_{3,5\cdot 3-1}$.

- Otherwise, let x be the largest member of I_1. Then $x \in \{1, 1+s, \ldots, 1+(d-1)s\} = \{1, 5, 9, 13, 17\}$. For if $21 \in I_1$, then $1, 2 \in I$, contradicting the assumption that I contains no consecutive labels. So there are $d = 5$ choices for I_1. Further, I_2 must be empty; otherwise, again, we have $1, 2 \in I$. Thus the remainder of I is isomorphic to an order ideal of $P_{s-2,d(s-2)-1} = P_{2,5\cdot 2-1}$.

To summarize, if I is an order ideal of $P_{s,ds-1}$ with no consecutive labels, then either I is isomorphic to an order ideal of $P_{s-1,d(s-1)-1}$, or I_1 has d options and the rest of I is isomorphic to an order ideal of $P_{s-2,d(s-2)-1}$. This proves (3.13).

From the above observation, we have the following characterization:

- Any order ideal of $P_{s,ds-1}$ with no consecutive labels is of the form $I = I_1 \cup \cdots \cup I_{s-1}$, where
  - Each I_k is either empty or of the form $\{k, k+s, \cdots, k+i_k s\}$, where $i_k \leq d-1$ if $k < s-1$, and $i_{s-1} < d-1$.
  - If I_k is nonempty, then I_{k+1} is empty.

In short: To make an order ideal, we hang strings of beads from the tops of the pillars in such a way the strings are not too long and adjacent pillars are not both decorated.

3.4.3 Computing the Generating Function

Our ultimate goal is to investigate the distribution of the size of $(s, ds-1)$-core partitions with distinct parts. To this end, we define the generating function

$$G_{d,s}(q) := \sum_{p} q^{|p|}, \tag{3.14}$$

where p ranges over $(n, ds-1)$-core partitions with distinct parts, and $|p|$ denotes the size of the partition p, i.e., the sum of its parts. We shall give an efficient scheme for computing $G_{d,s}(q)$ for fixed d and s.

Proceeding as in the previous section, we first compute the auxiliary generating function

$$F_{d,s}(q,t) := \sum_I q^{w(I)} t^{|I|}, \tag{3.15}$$

where I ranges over all order ideals of $P_{s,ds-1}$ with no consecutive labels; $w(I)$ is the sum of the labels in I; and $|I|$ is the number of labels in I. Then, as explained previously, we can obtain $G_{d,s}(q)$ by replacing occurrences of t^k in $F_{d,s}(q,t)$ with $q^{-k(k-1)/2}$.

To compute $F_{d,s}(q,t)$, we use the reasoning of the previous subsection, but this time we keep track of the weight of the order ideal.

First, we introduce yet another auxiliary generating function. For $1 \le k \le s-1$, let $P^k_{s,ds-1}$ be the sub-poset of $P_{s,ds-1}$ obtained by chopping off everything to the left of the kth column (note $P^1_{s,ds-1} = P_{s,ds-1}$). Define $F^k_{d,s}(q,t)$ as in (3.15), except I ranges over order ideals of $P^k_{s,ds-1}$ with no consecutive labels [note $F^1_{d,s}(q,t) = F_{d,s}(q,t)$].

By the reasoning of the previous section, the first column of an order ideal of $P^k_{s,ds-1}$ is either empty or of the form $\{k, k+s, \cdots, k+is\}$, where $0 \le i \le d-1$. Since the latter set has weight

$$q^{\sum_{j=0}^{i}(k+js)} t^{i+1} = q^{(i+1)(is/2+k)} t^{i+1},$$

we have the recursion

$$\begin{aligned} &F^k_{d,s}(q,t) = F^{k+1}_{d,s}(q,t) + \left(\sum_{i=0}^{d-1} q^{(i+1)(is/2+k)} t^{i+1}\right) F^{k+2}_{d,s}(q,t) \text{ for } 1 \le k \le s-2; \\ &F^{s-1}_{d,s}(q,t) = \sum_{i=0}^{d-2} q^{(i+1)(is/2+k)} t^{i+1}; \\ &F^s_{d,s}(q,t) := 1. \end{aligned} \tag{3.16}$$

Note that this is a recursion in the auxiliary index k, not in s and d.

Given s and d, we can use (3.16) to find $F^1_{d,s}(q,t) = F_{d,s}(q,t)$. Finally, we make the substitution $t^k \to q^{-k(k-1)/2}$ to find $G_{d,s}(q)$. All of this is done in the procedure `Gdn` in the Maple package `core2`. (See Appendix A.)

3.4.4 Distribution of the Size

Given fixed s and d, we can pick a uniform random $(s, ds-1)$-core partition with distinct parts, and consider its size, call it $X_{d,s}$. Then $X_{d,s}$ is a random variable, so it makes sense to inquire about its distribution. Since $G_{d,s}$ is the generating function for $X_{d,s}$, we can easily compute as many moments of the distribution as we please, for fixed s and d.

Using this information, we can investigate how the moments behave as functions of s and d. We will consider two cases: s is variable and d is fixed, and vice versa. In each case, we will consider the behavior of $X_{d,s}$ as the variable tends to infinity; in particular, we address the question of asymptotic normality. Finally, we will derive formulas for the first few moments as functions of *both* s and d.

First, we introduce some notation. Given a natural number k, let us denote

$$m_k(d,s) := \left[\left(q\frac{d}{dq}\right)^k G_{d,s}(q)\right]_{q=1}$$

to be the kth "pre-moment" of $X_{d,s}$. Define

$$M_k(d,s) := \frac{m_k}{G_{d,s}(1)} = \frac{m_k}{s_d(s)} = \mathbb{E}[X_{d,s}^k],$$

the kth (straight) moment of $X_{d,s}$. For example, the mean is $\mu_{d,s} = M_1$.

Denote the kth central moment by

$$M_k^c(d,s) := \mathbb{E}[(X_{d,s} - \mu)^k].$$

For example, the variance is $\sigma_{d,s}^2 := M_2^c$.

Finally, denote the kth standardized moment by

$$M_k^s(d,s) := \frac{M_k^c}{\sigma^k}.$$

Note that the central, straight, and standardized moments can easily be computed from the pre-moments.

Now, for numeric values of d and s, we can use our recursive scheme to easily compute all the quantities above. Analyzing the data for many values of d and s confirms the following:

Conjecture 3.14. *For each s, the kth pre-moment $m_k(d,s)$ of $X_{d,s}$ is a polynomial in d. Further, the degree of this polynomial is $2k + \lfloor s/2 \rfloor$.*

For example, our experimental evidence indicates that

$$
\begin{aligned}
\{\lim_{d\to\infty} M_k^s(d,3)\}_{k=3}^{\infty} &= 2/7\sqrt{5}, \frac{15}{7}, \frac{100}{77}\sqrt{5}, \frac{6625}{1001}, \frac{750}{143}\sqrt{5}, \ldots \\
&\approx .641, 2.14, 2.91, 6.62, 11.7, \ldots \\
\{\lim_{d\to\infty} M_k^s(d,4)\}_{k=3}^{\infty} &\approx .162, 2.08, 1.19, 6.20, 7.05, \ldots \\
\{\lim_{d\to\infty} M_k^s(d,5)\}_{k=3}^{\infty} &\approx .237, 2.22, 1.76, 7.43, 10.8, \ldots \\
\{\lim_{d\to\infty} M_k^s(d,6)\}_{k=3}^{\infty} &\approx .052, 2.36, .671, 7.80, 5.15, \ldots \\
&\ldots \\
\{\lim_{d\to\infty} M_k^s(d,10)\}_{k=3}^{\infty} &\approx -0.001, 2.62, .130, 10.1, 2.17, \ldots.
\end{aligned}
\tag{3.17}
$$

Recall that the standard normal distribution has standardized moments $0, 1, 0, 3, 0, 15, \ldots$. The sequences above seem to approach this as $s \to \infty$, leading us to the following:

Conjecture 3.15. *For each fixed s, the distribution of $X_{d,s}$ is* not *asymptotically normal as $d \to \infty$; that is, $(X_{d,s} - \mu_{d,s})/\sigma_{d,s}$ tends to some abnormal distribution X_s as $d \to \infty$. However, X_s* is *asymptotically normal; that is, $(X_s - \mu)/\sigma$ tends to the standard normal distribution as $s \to \infty$.*

Next, we fix d, and look at $X_{d,s}$ as a sequence of random variables indexed by s. The $d = 1$ case was already addressed previously, where we found that the pre-moments are given by polynomials in s and the Fibonacci numbers. In light of (3.13), we might expect the same to be true for arbitrary d, except we use the generalized Fibonacci numbers, $N_d(s)$:

Conjecture 3.16. *For each d, the kth pre-moment $m_k(d,s)$ of $X_{d,s}$ is of the form $a(s)N_d(s) + b(s)N_d(s+1)$, where a and b are polynomials in s.*

Again, experimental evidence verifies this claim. The one anomalous case seems to be $d = 2$, for which $N_d(s) = 2^{s-1}$. In this case, our methods do not yield nice formulas for the moments.

Upon computing the limits of the standardized moments, we do get the familiar sequence $0, 1, 0, 3, 0, 15, \ldots$ in this case, leading to the following:

Conjecture 3.17. *For each fixed d, the distribution of $X_{d,s}$ is asymptotically normal. That is, $(X_{d,s} - \mu_{d,s})/\sigma_{d,s}$ approaches the standard normal distribution as $s \to \infty$.*

Finally, it is possible (but computationally taxing) to obtain a single formula for the kth moment as a function of *both* k and s. The ideas is to fix *only* k and look at $\{m_k(d,s)\}_{s=2}^{\infty}$ as a sequence of polynomials in d. Further, due to Maple's ability to handle linear systems with symbolic coefficients, we can fit the data to the ansatz in Conjecture 3.16 , only now $a(s)$ and $b(s)$ will have coefficients which are rational functions of d:

Conjecture 3.18. *The kth pre-moment $m_k(d,s)$ of $X_{d,s}$ is of the form $A(s,d)N_d(s) + B(s,d)N_d(s+1)$, where A and B are degree $2k$ polynomials in s whose coefficients are rational functions in d.*

Due to the amount of data needed to fit the kth moment to the ansatz, it takes a few minutes even to generate the formula for the 3rd moment. Here, we present a small taste of the conjectures yielded by our Maple package. In the first two conjectures to follow, we could easily have presented formulas for many more moments, but we omit them to save space. See the Maple package `core2.txt` in Appendix A for more information.

In general, we conjectured that $M_k(d,s)$ is a rational function in d for s fixed. However, for $s = 3$ the straight moments seem to be polynomials:

Conjecture 3.19. *The expectation of $X_{d,3}$ is $d^2/3 + d/4 - 1/12$, and the variance is $4d^4/45 + d^3/12 - d^2/144 + d/24 + 31/720$.*

Here is an example in which we fix d.

Conjecture 3.20. *The expectation of $X_{3,s}$ is*

$$\frac{25}{39}\,s^2 - \frac{479}{507}\,s + \frac{406}{507} + \frac{N_3(s+1)}{N_3(s)}\,\left(-\frac{1}{39}s^2 + \frac{29}{169}\,s - \frac{158}{507}\right).$$

Finally, here is the expectation once and for all, in terms of both n and d:

Conjecture 3.21. *The expectation of $X_{d,s}$ is*

$$\frac{\left(5\,d^3+7\,d^2+d-1\right)s^2}{24(4\,d+1)}-\frac{\left(8\,d^5+21\,d^4+7\,d^3-d^2+3\,d-2\right)s}{24(16\,d^3-24\,d^2-15\,d-2)}$$
$$+\frac{17\,d^4+13\,d^3-9\,d^2-7\,d-2}{12(16\,d^3-24\,d^2-15\,d-2)}+\frac{N_d(s+1)}{N_d(s)}$$
$$\cdot\left(-\frac{\left(d^2-1\right)s^2}{24(4\,d+1)}-\frac{\left(2\,d^4-9\,d^3-16\,d^2-3\,d+2\right)s}{8(16\,d^3-24\,d^2-15\,d-2)}-\frac{d^4+20\,d^3+9\,d^2-20\,d-10}{12\,(d-2)\,(4\,d+1)^2}\right).$$

Note that this formula is singular at $d=2$, explaining the anomaly mentioned earlier. However, we can still make sense of the $d=2$ case by first plugging in a numeric value of s, (so that the N_d's become polynomials in d), then taking the limit as $d\to 2$. So this formula effectively works for *all* s and d.

Once again, many more results like these can easily be obtained using the Maple package `core2` in Appendix A. We invite you to try it for yourself.

3.5 Odd Parts and Other Restrictions

This section is adapted from our article [34].

It is both fascinating and frustrating that in enumeration problems, tweaking a problem ever so slightly turns it from almost trivial (and often, utterly trivial) to very difficult (and often, intractable). For example, it is utterly trivial that the number of n-step walks in the 2D rectangular lattice is 4^n, but just add the adjective "self-avoiding"—in other words, the number of such walks that never visit the same vertex twice—and the enumeration problem becomes (most probably) intractable and, at any rate, wide open.

Another example is counting permutations that avoid a pattern. The number of permutations, π, of length n that avoid the pattern 12 (i.e. you can't have $1\le i_1<i_2\le n$ such $\pi_{i_1}<\pi_{i_2}$) is trivially 1. A bit less trivially, but still very doable, is the fact that the number of permutations, π, of length n that avoid the pattern 123 (i.e. you can't have $1\le i_1<i_2<i_3\le n$ such $\pi_{i_1}<\pi_{i_2}<\pi_{i_3}$) is the good old Catalan number $(2n)!/(n!(n+1)!)$. But for most patterns, such an enumeration is (probably) intractable. The simplest wide open case, that we believe is intractable (but we would be happy to be proven wrong) is to count permutations that avoid the pattern 1324 (OEIS sequence A061552 [`https://oeis.org/A061552`]), for which the current record is knowing the 36 first terms.

Returning to the main topic, consider enumerating $(2s-1, 2s+1)$-core partitions into distinct parts. Armin Straub conjectured the deceptively simple formula 4^s. Alas, its (known) proofs are far from simple! Straub's conjecture was first proved, by Sherry H.F. Yan, Guizhi Qin, Zemin Jin, Robin D.P. Zhou [28], via an ingenious but rather complicated combinatorial proof. We provided a still non-trivial proof in Section 3.3, using "guess-and-check," and this was further simplified by Straub (see [32]). As far as we know, enumerating (s,t)-core partitions into distinct parts for other cases, say $(3s-1, 3s+1)$-core partitions, is wide open.

Leonhard Euler famously proved that the number of partitions of an integer n into *distinct* parts equals the number of partitions of the same n into *odd* parts. (This classical theorem was recently *refined* in a new, very surprising way, by Armin Straub [22].)

Moving on to counting $(s+1, s+2)$-core partitions into *odd* parts, it seems that the number of such partitions has nothing to do with the number of $(s+1, s+2)$-core partitions into *distinct* parts (i.e. F_{s+2}). This new problem seems (at least to us) much harder.

We will now describe our approach, its success (it enabled us to crank out 23 terms, thereby extending Straub's 11 terms, and with better computers, and more optimization, one may be able to crank out a few more terms), and its major shortcoming. At the end of the day, it is an *exponential time* (and memory!) algorithm.

3.5.1 Counting $(s+1, s+2)$-Core Partitions into Odd Parts

Again, we use the bijection between core partitions and posets of order ideals. Let $A_s := P_{s+1,s+2}$. A plot of this poset for $s = 9$ is shown in Figure 3.7.

Suppose S is an order ideal of A_s (i.e., S corresponds to an unrestricted $(s+1, s+2)$-core). Let i $(0 \leq i \leq n)$ be the smallest positive integer with the property that

$(s-1-i,i)$ is *not* a member of S: in other words, the smallest integer i such that $(s-1,0),(s-2,1),\ldots,(n-i,i-1)$ are members of S while $(s-1-i,i)$ is *not* a member of S. By our discussion preceding (3.5), we have a *canonical decomposition*

$$S \to (i, S_1, S_2), \quad 0 \le i \le n, \quad S_1 \in A_{i-2}, \quad S_2 \in A_{s-i} \tag{3.18}$$

that is obviously one-to-one.

Next, it is readily seen, by the mapping from order ideals to partitions,

$$(a_1, \ldots, a_k) \to (a_1 + k - 1, a_2 + k - 2, \ldots, a_k + 0),$$

that an order ideal of $P_{s+1,s+2}$ corresponds to an $(s+1, s+2)$-core partition into *odd* parts if and only if, when reading the occupied labels along diagonals, from bottom-right to top-left, starting from the rightmost diagonal and "walking" to the left, (i) the first label read is odd and (ii) the labels alternate in parity. For example, in Figure 3.7, the labels with red crosses comprise an order ideal corresponding to a partition with odd parts. Since only the parity matters, we can color the vertices of A_s by the colors "even" and "odd."

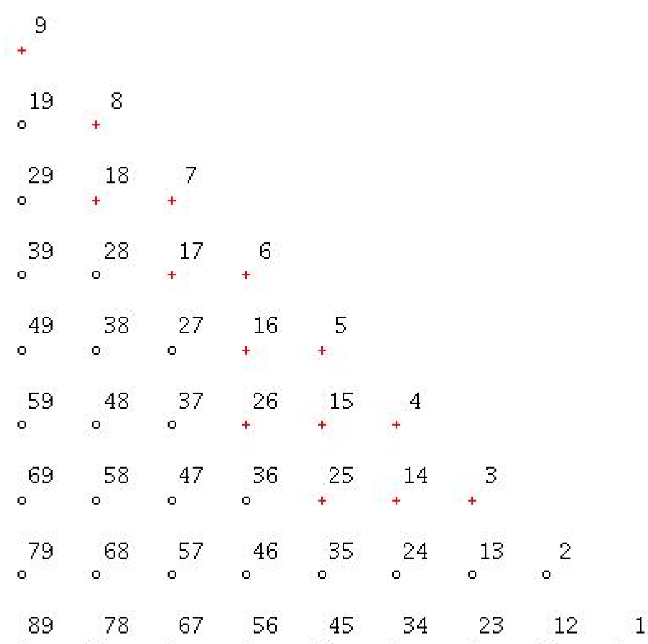

Figure 3.7: The lattice $A_9 := P_{10,11}$, with red crosses indicating an order ideal corresponding to a partition into odd parts.

Alas, one has to distinguish two cases. For both s even and odd, the label of $(s-1,0)$ is odd (since it is always 1), and as you proceed, in A_s along diagonals, the parities *alternate*. But for s odd, all the parities along the same row are the same, while if s is even, they alternate. Hence we are forced to consider the more general problem where there is a "coloring" parameter, let's call it c, ($c = 0$ or $c = 1$) such that the "color" of label (i, j) is

$$C(i,j) := 1 + ci + (1-c)j \pmod 2.$$

So let's forget, for now, about $(s+1, s+2)$-core partitions into odd parts, and instead define the following:

- Let $e^{(0)}(s)$ be the number of order ideals of $P_{s+1,s+2}$ such that when read along diagonals, the occupied vertices alternate in color using coloring parameter $c = 0$, and the first label is odd.

- Let $e^{(1)}(n)$ be the number of order ideals of $P_{s+1,s+2}$ such that when read along diagonals, the occupied vertices alternate in color using coloring parameter $c = 1$, and the first label is odd.

Once we find a way to compute both sequences $e^{(0)}_s$ and $e^{(1)}_s$, then our object of desire, the Straub sequence, enumerating $(s+1, s+2)$-core partitions into odd parts, let's call it n_s, is given by

$$n_s = \begin{cases} e^{(0)}_s \text{, if } s \text{ is even;} \\ e^{(1)}_s \text{, if } s \text{ is odd.} \end{cases}$$

3.5.2 Dynamical Programming

To characterize the sequences $e^{(0)}_s$ and $e^{(1)}_s$, we can try and extend the argument preceding the canonical decomposition (3.18) for counting *all* order ideals of $P_{s+1,s+2}$.

Suppose S is an order ideal of A_s whose labels satisfy the parity conditions. Let $(s-1-i, i)$ be the first unoccupied vertex of the order ideal S of A_s. Let (i, S_1, S_2) be its image under (3.18). The smaller order ideals S_1 (of A_{i-2}) and S_2 (of A_{s-i}) also have the property, that *within* each diagonal, the colors of the occupied vertices alternate, but, alas, as you move from one diagonal to the next one, the alternation may (and often does) break down. Also, the two components in the canonical decomposition are not "independent" but must satisfy some compatibility conditions.

This forces us to consider much more general creatures, order ideals whose "colors" (parity) alternate within each individual diagonal, and having, *additionally*, a given "coloring profile," the list of pairs of colors of the first and last vertices in each diagonal, reading from left to right. For example, the profile of the order ideal comprised of the red crosses in Figure 3.7 is $[[1,1],[0,0],[1,0]]$. There are three pairs in the profile since the order ideal is supported in the three outermost diagonals. The occupied vertices on the rightmost diagonal start with 3 and end with 9 (both odd); hence, the first pair is $[1,1]$. The lowest occupied vertex on the second diagonal has label 14 and the last one has label 18; hence for the second diagonal, we have $[0,0]$. Finally, the lowest label on the third diagonal is 25 and the highest is 26, hence $[1,0]$. Note that for any profile of an order ideal corresponding to an $(s+1,s+2)$-core partition

$$[[a_1,b_1],[a_2,b_2],\ldots,[a_k,b_k]],$$

b_i and a_{i+1} must have opposite parities. Also, $a_1 = 1$. We call such profiles "good profiles." Hence there are 2^{k-1} good profiles. Unfortunately, in order to use dynamical programming, we need to consider all 2^{2k} profiles for k diagonals (and it is easy to see that for us, $k \leq s/2$). Hence our algorithm is *exponential* in time (and memory).

We essentially use canonical decomposition (3.18) but refined to order ideals with a given profile, and at the end we sum over all good profiles.

The details are straightforward but rather tedious, and may be gotten from looking at the source code of the Maple package `OddArmin.txt` listed in Appendix A. See procedure `NuOIG(s,c)` giving $e_s^{(c)}$ for $c = 0$ and $c = 1$. It is obtained by adding up the outputs of procedure `NuOIP(s,c,P)` where `P` ranges over all "good profiles." This procedure works recursively using canonical decomposition (3.18), except now we have to keep track of profiles. For each "good" order ideal with given profile `P`, and for which $(s-1-i,i)$ is the "first" unoccupied vertex, the corresponding two smaller order ideals of A_{i-2} and A_{s-i} have implied profiles. Thus, `NuOIP(s,c,P)` sums `NuOIP(i-2,c,P1)*NuOIP(s-i,c,P2)` over all such "compatible" profile decompositions

`(P1,P2)` of the parent profile `P`.

The output was is as follows.

- The first 23 terms of the sequence $e_s^{(0)}$ (staring with $s = 0$) are

$$2, 4, 7, 17, 30, 80, 143, 404, 728, 2140, 3876, 11729, 21318, 65952, 120175, 378321,$$

$$690690, 2205168, 4032015, 13023324, 23841480, 77761008, 142498692. \quad \text{(A299294)}$$

- The first 23 terms of the sequence $e_s^{(1)}$ (staring with $s = 0$) are

$$2, 3, 7, 12, 31, 55, 152, 273, 790, 1428, 4271, 7752, 23767, 43263, 135221, 46675,$$

$$782968, 1430715, 4598804, 8414640, 27332956, 50067108, 164081764. \quad \text{(A299295)}$$

(Both sequences were brand new to the OEIS.)

But, we really don't care about $e_s^{(0)}$ when n is odd, or $e_s^{(1)}$ when s is even. We want the Straub sequence $e_s^{(s \mod 2)}$. In other words, we extract the even-indexed terms of the former sequence and the odd-indexed terms of the latter sequence, and then we *interleave* them. This yields the first 23 terms of the Straub sequence:

$$1, 2, 4, 7, 17, 31, 80, 152, 404, 790, 2140, 4271, 11729, 23767, 65952, 135221,$$

$$378321, 782968, 2205168, 4598804, 13023324, 27332956, 77761008. \quad \text{(A299293)}$$

This sequence was also not yet in the OEIS when we first discovered it.

But what about the "rejected" terms, the ones that we do not care about? Maybe we *should* care about them!

The first 23 terms of the sequence $e_n^{(s+1 \mod 2)}$ are

$$1, 2, 3, 7, 12, 30, 55, 143, 273, 728, 1428, 3876, 7752, 21318, 43263, 120175,$$

$$246675, 690690, 1430715, 4032015, 8414640, 23841480, 50067108. \quad \text{(A047749)}$$

To our utter surprise (and delight), this sequence *was* already in the OEIS (but for entirely different reasons!). It is sequence A047749 and has a very nice closed-form expression: If $s = 2m$, then $\frac{1}{2m+1} \cdot \binom{3m}{m}$, while if $s = 2m+1$, then $\frac{1}{2m+1} \cdot \binom{3m+1}{m+1}$. As mentioned in the OEIS entry, it is easily verified that its generating function, $Y = Y(x)$, satisfies the simple cubic equation

$$xY^3 - 2Y^2 + 3Y - 1 = 0.$$

We are almost sure that the generating function of the Straub sequence $n_s = e_s^{(s \mod 2)}$ also satisfies an algebraic equation, but the above 23 terms did not allow us to guess one.

Addendum: Paul Johnson's Discovery

In the first version of the paper from which this section was adapted, we said that we would gladly donate one hundred dollars to the OEIS, in honor of the first person to generate enough terms of the Straub sequence (A299293) that would enable the discovery of such an algebraic equation (with a few terms to spare, yielding a non-rigorous proof), and an additional one hundred dollars (either in honor of the same or different person(s) and/or machines), for a rigorous proof.

Soon after our paper appeared on the arXiv, Paul Johnson informed us that he had a proof that the "sister sequence" (A047749) counts $(s+1, s+2)$-core partitions with *even* parts. Even more impressively, he related the two sequences, implying a fast way to compute the original sequence we sought. In particular, one can derive a (rather complicated) algebraic expression for the generating function enumerating $(s+1, s+2)$-core partitions into odd parts. We would have needed 53 (as opposed to our 23) terms of the sequence to guess the generating function. At of the time of our writing this thesis, Johnson has a preliminary paper detailing his approach (which uses the abacus characterization of core partitions) on the arXiv [16]. We have compensated the OEIS as promised.

3.5.3 Enumerating Restricted Families of Core Partitions

Finally, we shall discuss some bonus families of partitions related to Straub's paper. In these cases, we were able to use symbolic computation to rigorously derive rational generating functions.

As noted, the sequence of numbers enumerating $(s+1, s+2)$-core partitions with distinct parts is $\{F_{s+2}\}_{s=0}^{\infty}$, whose generating function is the very simple rational function $\frac{1+x}{1-x-x^2}$. We shall now show that it is not hard to derive such rational generating functions to enumerate $(s+1, s+2)$-core partitions where each part gets repeated at most k times, for any, given, *specific* (i.e. numeric, not symbolic) k, where the former case corresponds to $k = 1$.

Again, consider the poset $A_s := P_{s+1,s+2}$, whose order ideals correspond to $(s+1, s+2)$-core partitions. Suppose S is an order ideal of A_s corresponding to a partition in which each part appears at most k times. This is equivalent to saying S contains at most k consecutive labels. (Note that, because S is an order ideal, a *necessary* condition for this is that the elements of S reside in the k outermost diagonals of A_s.)

As before, let $(s-1-i, i)$ be the smallest-labeled unoccupied point in the outermost diagonal of S, so that S contains the labels $1, \ldots, i$ but not $i+1$. Due to our new restriction, $i \leq k$. Again, let S_1 contain the elements of S below $(s-1-i, i)$ and not on the outer diagonal; let S_2 contain elements above $(s-1-i, i)$. Then S_1 is isomorphic to an arbitrary order ideal of A_{i-2}, and S_2 is isomorphic to an order ideal of A_{s-i} with no k consecutive labels.

So, with k fixed, we can see S as the "concatenation" of two types of order ideals—one with a filled-in base of size $\leq k$, and another of the same type as S. The generating function enumerating the first type of order ideals is a finite polynomial: its coefficients are Catalan numbers. So we obtain an algebraic equation satisfied by the desired generating function that can easily be solved in Maple. See the procedure `Fk` in the Maple package listed `core.txt` in Appendix A.

Here are the generating functions for $2 \leq k \leq 4$.

For $k = 2$:

$$-\frac{2x^2 + x + 1}{2x^3 + x^2 + x - 1},$$

whose first few coefficients are

$$1, 2, 5, 9, 18, 37, 73, 146, 293, 585, 1170, 2341, 4681, 9362, 18725, 37449, \ldots. \quad \text{(A077947)}$$

For $k = 3$:

$$-\frac{5x^3 + 2x^2 + x + 1}{5x^4 + 2x^3 + x^2 + x - 1},$$

whose first few coefficients are

$$1, 2, 5, 14, 28, 62, 143, 331, 738, 1665, 3780, 8576, 19376, 43837, 99265, \ldots. \quad \text{(A212340)}$$

For $k = 4$:

$$-\frac{14\,x^4 + 5\,x^3 + 2\,x^2 + x + 1}{14\,x^5 + 5\,x^4 + 2\,x^3 + x^2 + x - 1},$$

whose first few coefficients are

$$1, 2, 5, 14, 42, 90, 213, 527, 1326, 3317, 8022, 19608, 48272, 119073, \ldots. \quad \text{(A298367)}$$

For the generating functions, and first few terms, for the cases $5 \leq k \leq 20$, see the output file `oOddArmin3.txt` listed in Appendix A.

Inspired by the necessary condition mentioned above, let us enumerate $(s+1, s+2)$-core partitions into *odd* parts whose order ideals are restricted to the outer k diagonals.

As before, we classify S according to its profile P, a list of pairs, each pair giving the parities of the largest and smallest labels of S in a certain diagonal. Also, define $i(S)$ to be the smallest j such that $(j, 0)$ is occupied.

Call $(P(S), i(S))$ the "type" of S; for fixed k, there are finitely many types. Further, any S of a certain type is the concatenation of its elements on the x-axis with some smaller order ideal of A_{s-1} having a compatible type. Thus, the generating function of order ideals having a certain type satisfies some algebraic equation involving the generating functions of its "child" types. Once we solve this system and sum the generating functions over P, we get what we are after. See `Gk` in the Maple package `core.txt` listed in Appendix A.

For $k = 2$ the generating function is

$$-\frac{x^4 - x^3 - x^2 + x + 1}{x^5 - x^4 - 2\,x^3 + 3\,x^2 + x - 1},$$

whose first few coefficients are

$$1, 2, 4, 7, 15, 27, 56, 104, 210, 398, 791, 1517, 2988, 5769, 11306, \ldots. \quad \text{(A299099)}$$

For $k = 3$ the generating function is

$$-\frac{x^9 + x^8 - 4\,x^7 - 6\,x^6 + 8\,x^5 + 9\,x^4 - 5\,x^3 - 5\,x^2 + x + 1}{(x^9 + 2\,x^8 - 3\,x^7 - 9\,x^6 + 3\,x^5 + 14\,x^4 - x^3 - 7\,x^2 + 1)\,(x - 1)},$$

whose first few coefficients are

$$1, 2, 4, 7, 17, 31, 76, 144, 344, 670, 1560, 3103, 7079, 14315, 32152, \ldots . \quad \text{(A299102)}$$

For generating functions, and first few terms, for the cases $4 \le k \le 5$, see the output file `oOddArmin2.txt` listed in Appendix A.

Chapter 4

Inclusion-Exclusion and the Bonferroni Inequalities

In this chapter, we outline some "long shot" ideas from our research. The idea is to use computer implementations of inclusion-exclusion to improve existing results related to Ramsey numbers and Boolean satisfiability.

When applying the probabilistic method, one often needs to bound the probability of a union of events. A key tool for this is the *principle of inclusion-exclusion*:

Proposition 4.1 (Principle of Inclusion-Exclusion)**.** *Let $A_1, \ldots, A_N$ be events in a finite probability space. For $I \subset [N]$, define*

$$A_I = \bigcap_{j \in I} A_j.$$

Then,

$$\Pr[\cup_i A_i] = \sum_{i=1}^{N} (-1)^{i+1} \sum_{I \subset [N], |I|=i} \Pr[A_I].$$

Truncating the above sum at $i = m$ provides an upper or lower bound for the union, depending on the parity of m; thus we have the *Bonferroni inequalities*:

Proposition 4.2 (Bonferroni inequalities)**.** *With the notation of the previous Proposition, let $1 \leq m \leq N$. Then,*

$$\Pr[\cup_i A_i] \bowtie \sum_{i=1}^{m} (-1)^{i+1} \sum_{I \subset [N], |I|=i} \Pr[A_I],$$

where $\bowtie$ means $\leq$ if m is odd and $\geq$ if m is even.

Many proofs in the probabilistic method simply use the above inequality with $m = 1$; then, it simply says that the size of the union of sets is bounded from above by the sum of the cardinalities. For example, the famous Erdős-Szekeres bound on Ramsey numbers uses this fact [21]. But why stop at $m = 1$? Our goal here is to use computer methods to tighten the bounds and perhaps improve on existing results.

4.1 Bounding Ramsey Numbers

Recall that the kth diagonal *Ramsey number*, $R(k,k)$, is the smallest value of n such that any two-coloring of the edges of K_n (the complete graph on n vertices) is guaranteed to have monochromatic k-clique. Ramsey's theorem states that $R(k,k)$ is finite for each k and implies the upper bound $R(k,k) \leq C4^k$.

In 1947, Erdős proved an exponential *lower* bound on $R(k,k)$. His elegant proof marked the beginnings of the probabilistic method [21]. The idea is that $R(k,k) > n$ iff there exists a two-coloring of K_n with *no* monochromatic k-clique. Now, for n fixed, consider a random coloring of K_n in which each edge joining vertices in $[n] := \{1,\ldots,n\}$ is independently colored red or blue with equal probability. Let $A_1,\ldots,A_N$, where $N := \binom{n}{k}$, be the subsets of $[n]$ of size k. Define "bad events" $B_1,\ldots,B_N$, where B_i is the event that the A_i supports a monochromatic clique. Then

$$\Pr[B_i] = 2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}},$$

since B_i happens iff all $\binom{k}{2}$ edges supported on A_i are red, or all the edges are blue.

Now, let $P(n,k)$ be the probability that K_n, randomly colored as previously stated, has *no* monochromatic k-clique. Then, using Proposition 4.2 with $m=1$ (i.e., Boole's inequality)

$$\begin{aligned} P(n,k) &= 1 - \Pr \bigcup_i B_i \\ &\geq 1 - \sum_i \Pr[B_i] = 1 - \binom{n}{k} 2^{1-\binom{k}{2}}. \end{aligned}$$

When k and n are such that $P(n,k) > 0$, there exists a coloring of K_n with no monochromatic k-clique, and hence, $R(k,k) > n$. After some manipulations of the bound above, this gives rise to the lower bound $R(k,k) \geq C2^{k/2}$.

In theory, getting a tighter lower bound on $P(n,k)$ will improve the lower bound on $R(k,k)$. This inspired us to try to apply Proposition 4.2 with a higher (necessarily odd) value of m.

Our idea is implemented in the Maple package `ramsey` listed in Appendix A. The procedure `IncExc(n,k,m)` gives the bound on $P(n,k)$ using m steps of inclusion exclusion. However, due to the complicated nature of the intersections of the events $\{B_i\}$, we could only go up to $m = 3$. The procedure `RLB(k,m)` uses the previous one to compute a corresponding lower bound on $R(k, k)$; for example, if $m = 1$, it returns the Erdős bound.

Due to the complexity of the computations, we could only go up to $k = 23$. Table 4.1 shows a comparison of our $m = 3$ bounds with the Erdős $m = 1$ bounds. Initially, our bound actually is *worse*, but for $k \geq 21$, it seems to be an improvement. Also, as the plot in Figure 4.1 shows, the improvement seems to increase as k increases. Unfortunately, we do not have enough data to see *how* substantial our improvement is for large k.

k	Erdős lower bound on $R(k, k)$	Our bound
2	1	1
3	3	4
4	6	6
5	11	9
6	17	14
7	27	21
8	42	32
9	65	51
10	100	80
11	152	126
12	231	197
13	349	348
14	527	477
15	792	734
16	1186	1121
17	1771	1701
18	2639	2566
19	3923	3853
20	5817	5759
21	8609	8577
22	12715	12731
23	18747	18841
24	27595	27812
25	40557	40959
26	59522	60199

Table 4.1: Comparing our three-step inclusion-exclusion-based lower bound on diagonal Ramsey numbers with the one-step Erdős bound.

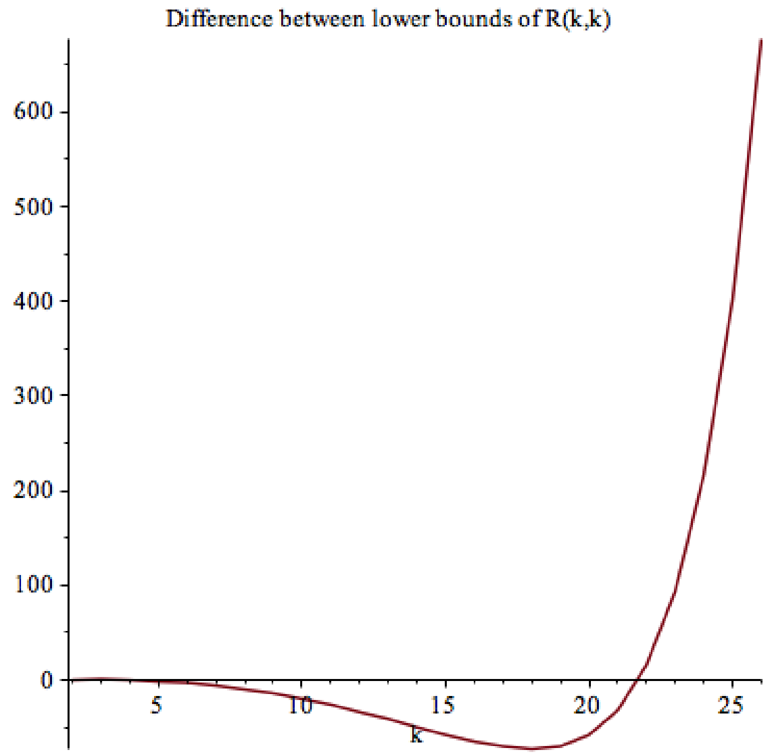

Figure 4.1: The difference between our our three-step inclusion-exclusion-based lower bound on diagonal Ramsey numbers and the one-step Erdős bound. The improvement seems to increase with k.

4.2 An Inclusion-Exclusion Based SAT Solver

Next, we use these ideas to approach the problem of Boolean satisfiability.

4.2.1 Introduction to SAT

First, some terminology. A *Boolean variable* is a variable which can take on values in $\{true, false\}$, or, equivalently, $\{0, 1\}$ (e.g. x). A *literal* is a Boolean variable or its negation (e.g. $\neg x$). *Disjunction* means "or" ($\vee$) and conjunction means "and" ($\wedge$). A *disjunctive clause* is a disjunction of literals (e.g. $x \vee \neg y \vee z$); similarly, we can define the *conjunctive clause.* A *conjunctive normal form* (CNF) is a conjunction of disjunctive clauses (e.g. $\neg z \wedge (y \vee z) \wedge (x \vee \neg y)$); similarly, we can define the *disjunctive normal form* (DNF).

We say that a CNF S in the variables $x_1, \ldots, x_n$ is *satisfiable* iff there exists an assignment of truth values to $x_1, \ldots, x_n$ that makes S true. For example, the CNF in the previous paragraph is satisfiable: The first clause forces $z = false$; then the second forces $y = true$; and the third forces $x = true$, giving us a valid assignment. On the other hand, the CNF $(x \vee y) \wedge \neg x \wedge \neg y$ is, of course, not satisfiable.

Given a CNF in n variables, one obvious way to determine its satisfiability is to check all 2^n assignments to the variables. There is an ongoing effort to develop more efficient algorithms to determine satisfiability. We call these algorithms "SAT solvers." Currently, even the most efficient SAT solvers are exponential time; one can always construct worst-case scenarios that take long for the algorithm to analyze. In fact, SAT was the first problem shown to be NP-complete, so a polynomial (in the size of the input) time SAT solver would indeed be breaking news [8].

Here, we shall certainly not present a polynomial-time algorithm, or even one that is practically more competent than current solvers. Rather, we wish to outline a simple, novel approach to solving SAT, analyze its strengths and weaknesses, and discuss how it might be used as the basis for a more powerful solver.

4.2.2 SAT and Inclusion-Exclusion

Suppose $S = C_1 \wedge \cdots \wedge C_N$ is a CNF with N clauses and n variables $x_1, \ldots, x_n$. Then, S is satisfiable iff $\neg S = \neg C_1 \vee \cdots \vee \neg C_N$ is not a tautology. So SAT can be rephrased as "given an arbitrary DNF, determine if it is a tautology." We shall use this formulation in our approach.

Thus, let $S = C_1 \vee \cdots \vee C_N$ be a DNF with N clauses and n variables $x_1, \ldots, x_n$. We wish to determine if all 2^n possible assignments to the variables result in S being true. We can interpret this probabilistically: If we pick a uniform random assignment, is $\Pr[S = true] = 1$? Equivalently, letting A_k be the event that C_k is satisfied, is $\Pr[\cup_k A_k] = 1$?

Using the notation of Proposition 4.1 (inclusion-exclusion), our problem amounts to finding $\Pr[A_I]$ for arbitrary $I \subset [N]$, which is easy: Let V be the set of literals appearing in the clauses $\{C_j : j \in I\}$; then, $\Pr[A_I] = 0$ if V contains a variable and its negation, and $\Pr[A_I] = 2^{-|V|}$ otherwise.

This idea is easily implemented to produce a simple inclusion-exclusion based SAT solver which always terminates with a correct answer. Such a solver, along with some test results, is briefly outlined in [18].

However, notice that the size of the inclusion-exclusion sum will grow with the number of clauses. Our idea is to use the Bonferroni inequalities (Proposition 4.2) to obtain a sequence of bounds on the "probability of satisfaction," formed by sequentially adding terms of the sum. The hope is that, in many cases, one does not actually need to compute the full sum before reaching a decision.

4.2.3 Details of the Algorithm

The method outlined above is implemented in the Maple package `sat`, listed in Appendix A.

We encode a DNF as a set of sets of integers: For example, `{{1,-2},{3}}` corresponds to $(x_1 \wedge \neg x_2) \vee x_3$. The `Merge` procedure is the equivalent of conjunction: `Merge({-1,2},{2,3})` returns `{-1,2,3}`, while `Merge({1,2},{-2,3})` returns `false` since these two clauses are "incompatible," i.e., not simultaneously satisfiable.

The main procedure is `Taut`. It inputs a DNF `S` and threshold `K`. We initialize `P=0` and `N=nops(S)`, the number of clauses. For `k` from 1 to `K`, we compute the `k`th term in the inclusion-exclusion sum and add it to `P`. For the sake of efficiency, a table is used to keep track of all *compatible* conjunctions of `k` clauses in `S`, so that at the `k`th stage, the table has at *most* `N` choose `k` entries. If we obtain a conclusive bound at some point in the loop, we return `[ans,k]`, where the first entry is `true` or `false`, depending on whether we found `S` to be a tautology. If we complete the whole loop without coming to a conclusion, we return `[P,k]`.

4.2.4 Testing the Solver

To test our solver, we use the procedure `RandNF(n,N,M)`, which generates a random DNF with `N` clauses in `n` variables, each containing `M` uniform random literals. By default, `M=3`, which we shall assume from now on.

The procedure `MetaTaut(n,N,K,M)` runs `Taut` on `M` random DNFs with `n` variables

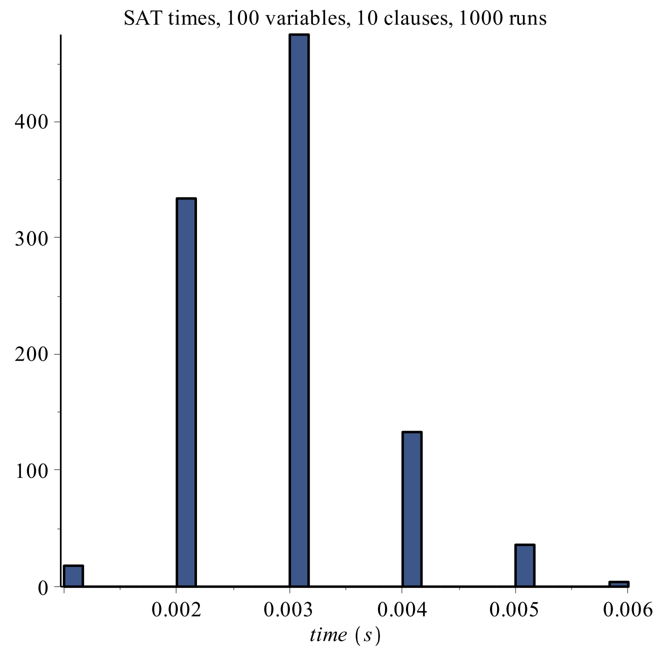

Figure 4.2: Histogram of runtimes of our SAT solver with low clause to variable ratio.

and `N` clauses and threshold `K`, and it records the run time and output of each trial.

The procedure `MetaTaut(n,N,K,M)` does the same, but instead of our solver, it uses Maple's built-in `tautology` procedure.

4.2.5 Runtimes

As one would expect, our solver seems to perform most competently when there are lots of variables but not too many clauses.

For example, Figure 4.2 shows a histogram of runtimes resulting from using `Taut` on 1000 random DNFs generated by `RandNF(100,10)`. In all of these cases, our solver arrived at the correct answer by the third step of the loop, and the longest runtime was .006s. As Figure 4.3 shows, the Maple solver performed slower in this case.

Further, we tested `Taut` on 10 random DNFs generated by `RandNF(1000,20)`, and it decided each of them was not a tautology by the seventh inclusion-exclusion step. The runtimes ranged from 2-58 minutes, with an average of 19. In this case, using `MapleTaut` resulted in an overflow error.

On the other hand, Figures 4.4 and 4.5 shows the results when 100 random DNFs generated by `RandTaut(100,20)` are used. Already, the number of clauses is enough to make our solver slower than Maple. In fact, in this case, only fifteen of the 100 random DNFs are solvable by `Taut` with threshold $k = 6$.

Also, we should point out that, in the situations where our method does seem promising, it seems that it almost always returns false. So, as it is, it probably has little practical use. Further, we are only testing it against a naive built-in Maple tautology function, rather than a sophisticated SAT solver.

4.2.6 Thresholds

Recall that, in `Taut(S,k)`, the argument `k` is the threshold, that is, the number of inclusion-exclusion summands computed before the procedure quits. Now, we investigate how the required threshold for a decisive answer is related to the number of variables `n` and number of clauses `N`.

The procedure `HowManyFinished(n,N,k,M)` runs `Taut` with threshold `k` on `M` random DNFs generated by `RandNF(n,N)`, and it outputs the proportion of conclusive runs. In other words, it estimates *success probability* that a DNF generated by `RandNF(n,N)` is solvable by our algorithm with threshold `k`.

Empirical evidence shows a phase shift behavior in the success probability if we fix n and k and vary N. Namely, there appears to be a critical number of clauses $N_c(n, k)$ at which the graph of the success probability has an inflection point. Of course, we have $N_c > k$, with N_c increasing in k.

Some plots exhibiting this phase shift are shown in Figure 4.6. Note that this behavior is reminiscent of the satisfiability phase shift studied in [27], where the behavior of the probability of a random CNF being satisfiable as a function of the ratio of the number of variables and clauses is studied.

4.3 SAT and the Lovász Local Lemma

Another powerful tool in the probabilistic method is the *Lovász local lemma*, used to determine if there is a positive probability that none of certain "bad events" occurs.

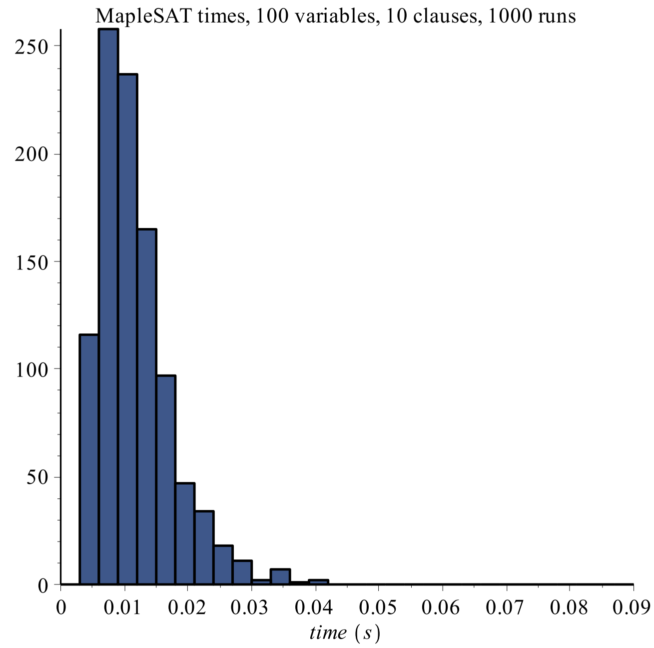

Figure 4.3: Histogram of runtimes of Maple's solver with low clause to variable ratio.

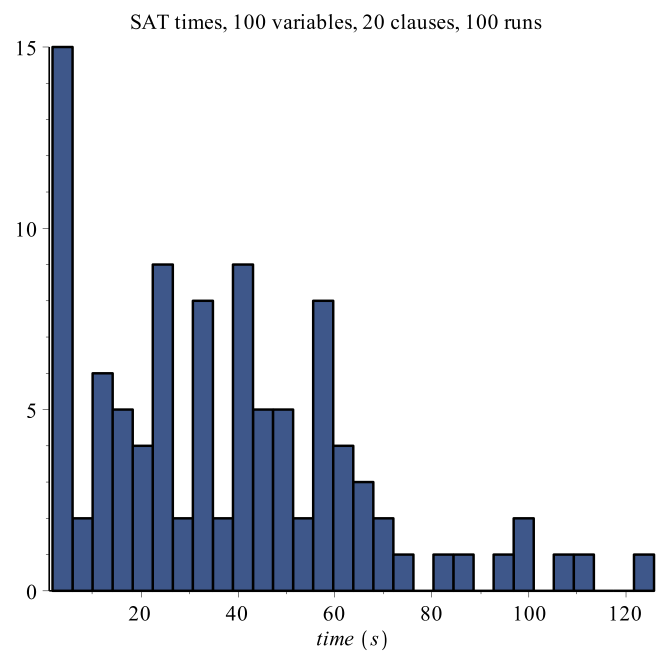

Figure 4.4: Histogram of runtimes of our SAT solver with higher clause to variable ratio.

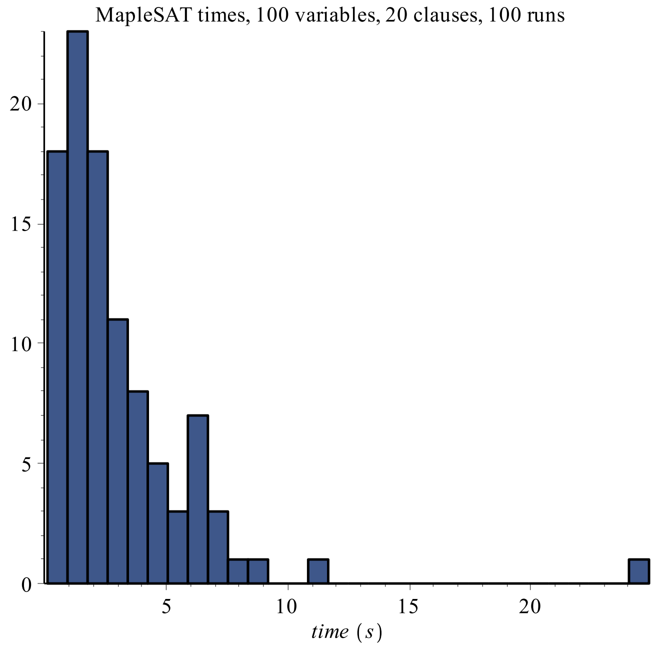

Figure 4.5: Histogram of runtimes of Maple's solver with higher clause to variable ratio.

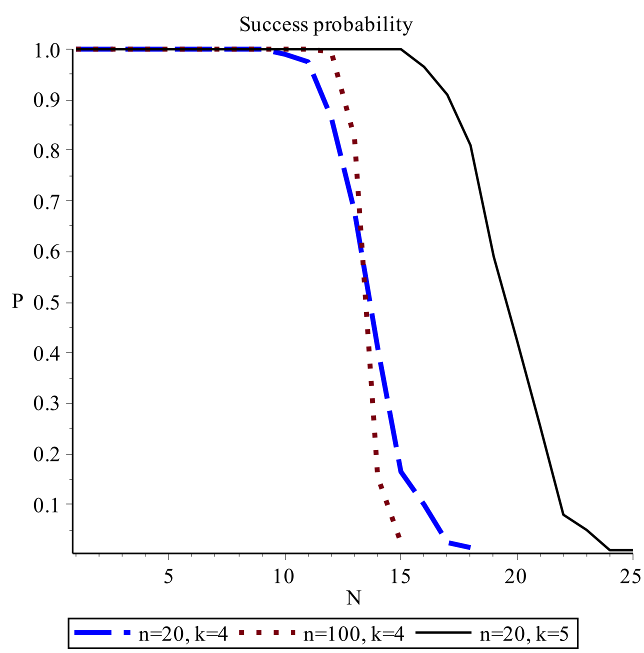

Figure 4.6: Here, n and N correlate with the number of variables and clauses, respectively; k is the threshold used in our solver; and P is the proportion of times our solver was successful, based on 200 runs with random DNFs.

Here, we will explain its application to SAT and, again, try to implement it with a computer.

4.3.1 Computerizing the Local Lemma

Given some "bad events" $\mathcal{A} = \{A_1, \ldots, A_N\}$, the Lovász local lemma can be used to verify that there is a positive probability that *none* of them occurs. Suppose G is a *dependency graph* on the vertex set $\mathcal{A}$: That is, events in $\mathcal{A}$ are mutually independent of their non-neighbors in G. Let $\Gamma(A)$ denote the neighborhood of A in G. Then the following holds:

Proposition 4.3 (Asymmetric Lovász local lemma)**.** *Suppose there exists a weight function* $x : \mathcal{A} \to [0, 1)$ *such that*

$$\forall A \in \mathcal{A}, \quad \Pr(A) \leq x(A) \prod_{B \in \Gamma(A)} (1 - x(B)).$$

Then $\Pr(\bigcap_i A_i^c) > 0$*.*

In applications, the weight function $x(A)$ is usually found ad hoc. If we assume each vertex of the dependency graph has degree $\leq d$ and set $x \equiv 1/(d + 1)$, we obtain the following:

Proposition 4.4 (Symmetric Lovász local lemma)**.** *Suppose each event* A_i *satisfies* $\Pr(A_i) \leq p$ *and is independent of all but at most* d *of the other events. If*

$$ep(d + 1) \leq 1,$$

then $\Pr(\bigcap_i A_i^c) > 0$*.*

The procedure `LLLs(P,G)` in the Maple package checks if the events A_i satisfy the symmetric local lemma, where the lists P and G satisfy $P[i] = \Pr(A_i)$ and $G[i] = \{j : A_j \in \Gamma(A_i)\}$.

Computerizing the asymmetric local lemma is harder, since, as far as we know, there is no systematic and efficient way to look for a valid weight function x. Somewhat arbitrarily, the procedure `LLL(P,G)` uses the weight function $x(A) = 1/(|\Gamma(A)| + 1)$. The motivation for this choice is that, when the dependency graph is uniform, it reduces to the symmetric local lemma.

4.3.2 Applying the Local Lemma to SAT

The article [14] addresses a theoretical application of the local lemma to SAT, focusing on using it to derive combinatorial conditions for the satisfiability of CNFs. Here, we present a computer application of the local lemma to SAT.

Let us return to the setup used previously. We have a DNF $S = C_1 \vee \cdots \vee C_N$ with variables $x_1, \ldots, x_n$, which are assigned true/false values uniformly at random. We let A_k be the event that C_k is true. Then S is *not* a tautology iff there is a positive probability that *none* of the events A_k occurs. So we can apply the local lemma.

We form a dependency graph G by joining A_i and A_j iff the clauses C_i and C_j have common variables. Also, $\Pr(A_i) = 2^{-n_i}$, where n_i is the number of literals in C_i; for example, for 3-SAT, $\Pr(A_i) = 1/8$.

The procedure `DNFtoPG(S)` converts the DNF S to a pair P, G, which can be passed to one of the `LLL` procedures. If the procedure returns *true*, then we can conclude that S was *not* a tautology; otherwise, this method is inconclusive.

Unfortunately, `LLLs` rarely succeeds at detecting a non-tautology, and `LLL` is only slightly better. For example, out of 100 non-tautologies generated by `RandNF(100,10)`, only 26 were detected by `LLLs` and 37 by `LLL`. Out of 100 non-tautologies generated by `RandNF(100,15)`, only 2 were detected by `LLLs` and 3 by `LLL`. We expect that this is due to the behavior of the dependency graph. It would be interesting to develop a "clever" asymmetric local lemma algorithm that tailors the weight function to work for the given dependency graph.

Chapter 5

A Boolean Analogue of Integer Covering Systems

This chapter is adapted from our paper [33].

Our experimentation with Boolean functions led us to next consider their relation with integer covering systems. We will now explain how we were able to define analogs of *exact* and *distinct* integer covering systems for Boolean tautologies, and come up with some new results and conjectures.

5.1 Integer Covering Systems

5.1.1 Prime Numbers are Sometimes Red Herrings

The great French mathematical columnist Jean-Paul Delahaye [9] recently posed the following brain-teaser, adapting a beautiful puzzle, of unknown origin, popularized by Peter Winkler in his wonderful book [25] (pp. 35-43).

Here is a free translation from the French.

One places nine beetles on a circular track, where the nine arc distances, measured in meters, between two consecutive beetles are the first nine prime numbers, 2,3,5,7,11,13,17,19 and 23. The order is arbitrary, and each number appears exactly once as a distance.

At starting time, each beetle decides *randomly* whether she will go, traveling at a speed of 1 meter per minute, clockwise or counter-clockwise. When two beetles bump into each other, they immediately do a “U-turn,” i.e. reverse direction. We assume that the size of the beetles is negligible. At the end of 50 minutes, after many collisions, one notices the distances between the new positions of the beetles. The nine distances are exactly as before, the first nine prime numbers! How to explain this miracle?

Before going on, we invite you to solve this lovely puzzle all by yourself.

Here is the solution. Note that the length of the circular track is $2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 = 100$ meters.

Let each beetle carry a flag, and whenever they bump into each other, let them exchange flags. Since the flags always move in the same direction, and also move at a speed of 1 meter per minute, after 50 minutes, each flag is *exactly* at the "antipode" of its original location; hence, the distances are the same! Of course, this works if the original distances were *any* sequence of numbers: All that they have to obey is that their sum equals 100, or more generally, that half the sum of the distances divides the product of the speed (1 meter per minute in this puzzle) and the elapsed time (50 minutes in this puzzle).

This variation, due to Delahaye, is *much* harder than the original version posed in [25], where also the initial distances were arbitrary. In Delahaye's rendition, the solver is bluffed into trying to use the fact that the distances are primes. Something analogous happened to the great Paul Erdős, the patron saint of combinatorics and number theory, who introduced *covering systems*.

5.1.2 Erdős' Covering Systems

In 1950, Paul Erdős [11] introduced the notion of covering systems. A *covering system* is a finite set of arithmetical progressions

$$\{a_i \pmod{m_i} : 1 \le i \le N\},$$

whose union is the set of all non-negative integers. For example

$$\{0 \pmod{1}\},$$

is such a (not very interesting) covering system, while

$$\{0 \pmod{2}, 1 \pmod{2}\},$$

and

$$\{0 \pmod{5}, 1 \pmod{5}, 2 \pmod{5}, 3 \pmod{5}, 4 \pmod{5}\}$$

are other, almost as boring examples. A slightly more interesting example is

$$\{0 \pmod 2, 1 \pmod 4, 3 \pmod 4\}.$$

A covering system is *exact* if all the congruences are disjoint (like in the above boring examples). It is *distinct* if all the moduli are different.

Erdős gave the smallest possible example of a distinct covering system [From now on, let $a\,(b)$ mean $a \pmod b$]:

$$\{0\,(2), 0\,(3), 1\,(4), 5\,(6), 7\,(12)\}.$$

Of course, the above covering system is not exact since, for example, $0\,(2)$ and $0\,(3)$ both contain any multiple of 6. A theorem proved by Mirsky and (Donald) Newman, and independently by Davenport and Rado (described in [12]) implies that a covering system cannot be both exact *and* distinct. Even a stronger statement holds. Assuming that our system $\{a_i\,(m_i)\}_{i=1}^{N}$ is written in non-decreasing order of the moduli $m_1 \leq m_2 \leq \cdots \leq m_N$, the Mirsky-Newman-Davenport-Rado theorem asserts that $m_{N-1} = m_N$; in other words, the two top moduli are equal (and hence an exact covering system can never be distinct). See [36] for an exposition of their snappy proof. While their proof was nice, it was not as nice as the combinatorial-geometrical proof that was found by Berger, Felzenbaum, and Fraenkel [6], [5], and exposited in [36]. In fact, they proved the more general Znam theorem that asserts that the highest moduli shows up at least p times, where p is the smallest prime dividing $\operatorname{lcm}(m_1, \ldots, m_N)$. Jamie Simpson [20] independently found a similar proof, but unfortunately chose not to express it in the evocative geometrical language.

5.1.3 The Berger-Felzenbaum Revolution: From Number Theory to Discrete Geometry via the Chinese Remainder Theorem

While it is a sad truth that the set of positive integers is an *infinite* set, a covering system is a *finite* object. In order to verify that a covering system, $\{a_i\,(m_i)\}_{i=1}^{N}$, is indeed one, it suffices to check that it covers all the integers n between 0 and $M-1$, where

$$M = \mathrm{lcm}\,(m_1, m_2, \ldots, m_N).$$

By the fundamental theorem of arithmetic

$$M = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k},$$

where $p_1, \ldots, p_k$ are primes and $r_1, \ldots, r_k$ are positive integers.

For the sake of simplicity, let's assume that M is square-free; i.e., all the exponents $r_1, \ldots, r_k$ equal 1. The same reasoning, only slightly more complicated, applies in the general case.

Now we have

$$M = p_1 p_2 \cdots p_k.$$

The ancient, but still useful, *Chinese Remainder theorem* tells you that there is a bijection between the set of integers between 0 and $M-1$, let's call it $[0, M-1]$, and the Cartesian product of $[0, p_i - 1]$, $i = 1 \ldots k$:

$$f := [0, M-1] \to \prod_{i=1}^{k} [0, p_i - 1],$$

defined by

$$f(x) := (x \pmod{p_1}, \ldots, x \pmod{p_k}).$$

So each integer in $[0, M-1]$ is represented by a *point* in the $p_1 \times p_2 \times \cdots \times p_k$ k-dimensional discrete box $\prod_{i=1}^{k} [0, p_i - 1]$.

Suppose $a\,(m)$ is a member of our covering system. Since m is a divisor of M, it can be written as a product of some of the primes in $\{p_1, \ldots, p_k\}$, say

$$m = p_{i_1} p_{i_2} \cdots p_{i_s}.$$

Let

$$m_{i_1} = a \pmod{p_{i_1}}, \ldots, m_{i_s} = a \pmod{p_{i_s}}.$$

It follows that the members of the congruence $a\,(m)$ correspond to the points in the $k-s$-dimensional *sub-box*

$$\{(x_1, \ldots, x_k) \in [0, p_1 - 1] \times \cdots \times [0, p_k - 1] : x_{i_1} = m_{i_1}, \ldots, x_{i_s} = m_{i_s}\}.$$

For example if $M = 30 = 2 \cdot 3 \cdot 5$, the congruence class $7\,(10)$ corresponds to the one-dimensional subbox

$$\{\,(x_1, x_2, x_3) : x_1 = 1, 0 \leq x_2 \leq 2, x_3 = 2\},$$

since $7 \mod 2 = 1$ and $7 \mod 5 = 2$. In other words a covering system (with square-free M) is nothing but a way of expressing a certain k-dimensional discrete box as a union of sub-boxes. This was the beautiful insight of Marc Berger, Alex Felzenbaum, and Aviezri Fraenkel, nicely exposited in [36].

5.1.4 Erdős's Famous Problem and Bob Hough's Refutation

Erdős [12] famously asked whether there exists a distinct covering system

$$a_i \pmod{m_i}, \quad 1 \leq i \leq N, \quad m_1 < m_2 < \cdots < m_N,$$

with the smallest modulo, m_1, arbitrarily large.

As computers got bigger and faster, people (and their computers) came up with examples that progressively made m_1 larger and larger, and many humans thoughts that indeed m_1 can be made as large as one wishes. This was brilliantly refuted by Bob Hough [15] who proved that $m_1 \leq 10^{16}$. This is definitely not sharp, and the true largest m_1 is probably less than 1000.

5.2 Boolean Functions

Let's now move on from number theory to something apparently very different: logic!

First, recall some basic definitions. A *Boolean function* (named after George Boole [7]) of n variables is a function from $\{False, True\}^n$ to $\{False, True\}$. Altogether there are 2^{2^n} Boolean functions of n variables. Any Boolean function $f(x_1, \ldots, x_n)$ is determined by its *truth table*, or equivalently, by the set $f^{-1}(True)$, one of the 2^{2^n} subsets of $\{False, True\}^n$.

The simplest Boolean functions are the *constant* Boolean functions *True* (the *tautology*) corresponding to the whole of $\{False, True\}^n$, and *False* (the *anti-tautology*) corresponding to the empty set.

In addition to the above constant Boolean functions, there are three *atomic* functions. The simplest is the *unary* function **NOT**, denoted by $\bar{x}$, that is defined by

$$\bar{x} = \begin{cases} False\,, & if \quad x = True; \\ True\,, & if \quad x = False\,. \end{cases}$$

The two other fundamental Boolean functions are the (inclusive) **OR**, denoted by $\vee$ and **AND**, denoted by $\wedge$. $x \vee y$ is True unless both x and y are false, and $x \wedge y$ is true only when both x and y are true.

By iterating these three operations on n variables, one can get many *Boolean expressions*, and each Boolean function has many possible expressions.

From now on we will denote, as usual, *true* by 1 and *false* by 0. Also let $x^1 = x$ and $x^0 = \bar{x} = 1 - x$.

One particularly simple type of expression is a *conjunction* (also called *term*). It is anything of the form, for some t, called its *size*,

$$x_{i_1}^{j_1} \wedge \cdots \wedge x_{i_t}^{j_t} \quad ,$$

where $1 \leq i_1 < \cdots < i_t \leq n$ and $j_i \in \{0, 1\}$ for all $1 \leq i \leq t$.

Of interest to us in this article is the type of expression called the *Disjunctive Normal Form* (DNF). It simply has the form

$$\bigvee_{i=1}^{N} C_i,$$

where each C_i are pure conjunctions. Its dual is the *Conjunctive Normal Form* (CNF).

Every Boolean expression corresponds to a unique function, but every function can be expressed in many ways, and even in many ways that are DNF. One way that is the most straightforward way is the *canonical DNF* form

$$\bigvee_{\{v \in f^{-1}(1)\}} \bigwedge_{i=1}^{n} x_i^{v_i}.$$

Note that a pure conjunction of length t

$$x_{i_1}^{j_1} \wedge \cdots \wedge x_{i_t}^{j_t}$$

corresponds to a *sub-cube* of dimension $n-t$, namely to

$$\{(x_1, \ldots, x_n) : x_{i_1} = j_1, \ldots, x_{i_t} = j_t\}.$$

Hence, one can view a DNF as a (usually not exact) *covering* of the set $f^{-1}(1)$ of truth-vectors by sub-cubes. In particular, a *DNF tautology* is a covering of the whole n-dimensional unit cube by lower-dimensional sub-cubes.

DNFs and the Million Dollar Problem: The most fundamental problem in theoretical computer science, the question of whether **P** is *not* **NP** (of course it is not, but proving it rigorously is another matter), is equivalent to the question of whether there exists a polynomial time algorithm that decides if a given Disjunctive Normal Form expression is the tautology (i.e. the constant function 1). Of course, there is an obvious *algorithm*: For each term, find the truth-vectors covered by it, take the union, and see whether it contains all the 2^n members of $\{0,1\}^n$. But this takes *exponential time* and *exponential memory.*

The Covering System Analog: Input a system of congruences

$$a_i \pmod{m_i}, \quad 1 \leq i \leq N,$$

and decide, in *polynomial time*, whether it is a covering system. Initially it seems that we need to check infinitely many cases, but of course (as already noted above), it suffices to check whether every integer between 1 and $\mathrm{lcm}\,(m_1, \ldots, m_N)$ belongs to at least one of the congruences. This seems fast enough! Alas, the size of the input is the sum of the number of digits of the a_i's and m_i's and this is less than a constant times the *logarithm* of *lcm* $(m_1, \ldots, m_N)$, so just like for Boolean functions, the naive algorithm is exponential time (and space) in the *input size.*

5.2.1 Boolean Analogs of Covering Systems

We next consider Boolean function analogs of covering systems. The first one to consider such analogs was Melkamu Zeleke [41]). Here we continue his pioneering work.

We saw that a DNF tautology is nothing but a covering of the n-dimensional unit cube $\{0,1\}^n$ by sub-cubes. So it is the analog of a covering system.

The analog of *exact* covering systems is obvious: all the terms should cover disjoint sub-cubes. For example, when $n = 2$, (from now on xy means $x \wedge y$)

$$x_1x_2 \vee x_1\bar{x}_2 \vee \bar{x}_1x_2 \vee \bar{x}_1\bar{x}_2,$$

$$x_1 \vee \bar{x}_1x_2 \vee \bar{x}_1\bar{x}_2,$$

are such.

In order to define *distinct* DNF, we define the *support* of a conjunction as the set of the variables that participate. For example, the support of the term $\bar{x}_1\bar{x}_3x_4x_6$ is the set $\{x_1, x_3, x_4, x_6\}$. In other words, we ignore the negations. For each t-subset of $\{x_1, \ldots, x_n\}$ there are 2^t conjunctions with that support. Geometrically speaking, two terms with the same support correspond to sub-cubes which are "parallel" to each other.

Note that the supports correspond to the modulus, m, and the assignments of negations (or no negation) corresponds to a residue class modulo m. Thus, we say a DNF tautology is *distinct* if it has distinct supports.

The Boolean analog of the Mirsky-Newman-Davenport-Rado theorem is almost trivial. First, suppose we have an exact DNF tautology where the largest support has size n. That corresponds to a point (a 0-dimensional subcube). If it is the only one, then since a conjunction of length t covers 2^{n-t} points, if all the other ones are strictly smaller than n, and since they are all disjoint, they cover an even number of points, hence there is no way that an exact DNF tautology would only have one term of size n.

If the largest size of a term is $< n$, then by projecting on appropriate sub-boxes one can reduce it to the former case, and see that it must have a mate.

5.2.2 The Boolean Analog Erdős' Minimum Modulus Question

An obvious example of a distinct DNF tautology in n variables is

$$\bigvee_{i=1}^{n} x_i \ \vee \ \wedge_{i=1}^{n} \bar{x}_i.$$

More generally, for every $1 \le t \le n$, $(t \neq n/2)$ the following is a distinct DNF tautology:

$$\left(\bigvee_{1 \le i_1 < i_2 < \cdots < i_t \le n} x_{i_1} \cdots x_{i_t} \right) \vee \left(\bigvee_{1 \le j_1 < j_2 < \cdots < j_{n-t} \le n} \bar{x}_{j_1} \cdots \bar{x}_{j_{n-t}} \right).$$

This follows from the fact that by the pigeon-hole principle, every 0-1 vector of length n either has $\ge t$ 1's or $\ge n - t$ 0's.

Taking n to be odd, the above DNF tautology with $t = (n-1)/2$ has "minimal moduli" (supports) of size $(n-1)/2$, and that can be made as large as one wishes. So in the Boolean case, the answer to Erdős' question "can the minimum modulus (or rather, support size) be made arbitrarily large" is *yes*!

5.2.3 First Challenge

This leads to a more challenging problem: For each specific n, how large can the minimum clause size, let's call it k, in a distinct DNF tautology, be?

An obvious *necessary condition*, on density grounds, is that

$$\sum_{i=k}^{n} \binom{n}{i} \frac{1}{2^i} \ge 1. \tag{5.1}$$

(Each subset of size i of $\{1, \ldots, n\}$ can only show up once and covers 2^{n-i} vertices of the n-dimensional unit cube. Now use Boole's inequality that says that the number of elements of a union of sets is $\le$ than the sum of their cardinalities).

Let A_n be the largest such k satisfying (5.1). The first 14 values of A_n are

$$1, 1, 1, 2, 3, 4, 4, 5, 6, 7, 7, 8, 9, 10 \ldots.$$

We were able to find such optimal distinct DNF tautologies (i.e., with smallest clause size A_n) for all $n \le 14$ except for $n = 10$, where the best that we came up with was one that covers 1008 out of the 1024 vertices of the 10-dimensional unit cube, leaving 16 points uncovered, and for $n = 14$, where 276 out of the $2^{14} = 16384$ points were left uncovered.

See the output file `odt2.txt` listed in Appendix A.

5.2.4 Second Challenge

Another challenge is to come up with distinct DNF tautologies with all the terms of the *size*. By density arguments a necessary condition for the existence of such a distinct DNF tautology

$$\binom{n}{m}\frac{1}{2^m} \geq 1.$$

Let B_m be the largest such m. The first 14 values are

$$0, 0, 1, 2, 3, 3, 4, 5, 6, 6, 7, 8, 9, 9 \ldots.$$

Obviously for $n = 3$, where $B_3 = 1$, it is not possible, since $x_1 \vee x_2 \vee x_3$ can't cover everything. We were also unable to find such optimal DNF tautologies for $n = 5$, where $B_5 = 3$ and we had to leave one vertex uncovered, $n = 9$, (with $B_9 = 6$), where 13 vertices were left uncovered, and $n = 13$ (with $B_{13} = 9$) where $2^{13} - 8090 = 102$ vertices were left uncovered. For the other cases with $n \leq 14$, we met the challenge. See the output file `odt1.txt` listed in Appendix A. You are also welcome to experiment for yourself with the package `dt.txt`!

Appendix A

Accessing the Supplemental Computer Material

This thesis references Maple packages and computer-generated data that can be found on the author's site, http://sites.math.rutgers.edu/~az202/, and/or Dr. Zeilberger's site,http://sites.math.rutgers.edu/~zeilberg/.

All packages and data are text files. They can be read into Maple by saving them in the working directory and executing read(<file_name>); at the Maple prompt.

Table B.1 lists URLs for specific materials mentioned. In some cases, we list a link to the "front" of one of our articles, which in turn links to the relevant materials.

Keyword	URL
Armin	http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/armin.html
core	http://sites.math.rutgers.edu/~zeilberg/tokhniot/core.txt
core2	http://sites.math.rutgers.edu/~az202/Z/core/core2.txt
dt	http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/dt.html
Feller	http://sites.math.rutgers.edu/~az202/Z/Feller.txt
OddArmin	http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/oddarmin.html
odt1	http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/dt.html
odt2	http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/dt.html
oOddArmin2	http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/oddarmin.html
oOddArmin3	http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/oddarmin.html
ramsey	http://sites.math.rutgers.edu/~az202/Z/ramsey/ramsey.txt
sat	http://sites.math.rutgers.edu/~az202/Z/sat/sat.txt
theorems	http://sites.math.rutgers.edu/~az202/Z/core/theorems.txt

Table A.1: Supplemental computer materials.

Appendix B

Index of Notation

Here, we list terms and notation that are potentially unfamiliar or specific to this thesis. The page reference gives the first occurrence of the term.

Term	Description	Page
$a_1(w)$	number of losing times	5
$a_2(w)$	number of break-even times	5
$a_3(w)$	last break-even time	6
$a_3(w)$	number of sign changes	6
A_s	the poset $P_{s,s+1}$	41
CNF	conjunctive normal form	53
distinct	(covering system)	64
distinct	(tautology)	69
DNF	disjunctive normal form	53
$e^{(0)}(s)$	a sequence	43
$e^{(1)}(s)$	another sequence	43
$EO(a,b)$	a poset	26
exact	(covering system)	64
exact	(tautology)	69
$G_{d,s}(q)$	g.f. of $(s, ds-1)$-cores with distinct parts	35
$G_s(q)$	g.f. of $(s, s+1)$-cores with distinct parts	16
$l(w)$	length of walk w	5
moment	(straight, central, standard)	7
$N_d(s)$	generalized Fibonacci number	33
$OE(a,b)$	a poset	27
order ideal	(of a poset)	24
partition	(core, hook length)	14
pre-moment	moment before normalizing	37
P_s	$(s, s+1)$-cores with distinct parts	16
$P_{s,t}$	poset whose order ideals $\leftrightarrow$ (s,t)-cores	24
$R(k,k)$	diagonal Ramsey number	51
$S_s(q)$	a Straub polynomial	29
W	up/right walks	6
$W_{n,n}$	walks to (n,n)	6
W^S	walks with step set S	10

Table B.1: Index of notation.

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