SYMBOLIC COMPUTATION TO STUDY EXPLICIT GRÖBNER BASES AND LATTICE PATH ENUMERATION

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ABSTRACT OF THE DISSERTATION

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Experimental mathematics involves using computation and algorithms to study mathematical objects, typically with computer-assisted proving. This dissertation demonstrates experimental methods in researching various problems.

The first project expands upon Haglund, Rhoades, and Shimonozo's work on finding the reduced Gröbner basis of the ideal of elementary symmetric polynomials in n variables of degree d for $d = n - k + 1, \ldots, n$. Using symbolic computation and experimentation, we construct the reduced Gröbner basis for the ideal generated by the elementary symmetric polynomials in n variables of arbitrary degrees.

The remaining projects focus on Dyck, Motzkin, and similar paths. Using Zeilberger's automated procedures to find the weight enumerator for specific families of restricted Dyck paths, we extend these findings to infinite families through grammatical proofs. We then generalize the procedures to find the weight enumerator for restricted Motzkin paths. The next project explains how to automatically generate the weight enumerator of generalized Dyck paths, i.e. paths in the $xy-$plane from $(0, 0)$ to $(n, 0)$ with an arbitrary set of atomic steps that never go below the $x-$axis. Expanding on this, we compute the generating functions for the sum of the areas

under such paths as well as the sum of a given power of the areas.

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Contents

Introduction

This dissertation focuses on the application of experimental mathematics to a number of problems, demonstrating its methodology. In the most general sense, experimental mathematics is the methodology of using computation and algorithms to study mathematical objects, typically involving computer-assisted proving. Programming and developing algorithms are important tools in generating ideas and tackling potential problems. While mathematical investigation is traditionally done by hand, the methodology of experimental mathematics has shown itself to be invaluable. Computers can provide mathematicians with a vast amount of data used to form new conjectures, which would be extremely tedious – if not impossible – to obtain by hand. As computers become increasingly powerful and allow more flexibility in programming, mathematicians need to adapt and develop new research methods. By designing algorithms, mathematicians can "teach" computers to form and rigorously prove conjectures, unlocking their untapped potential in furthering mathematical studies.

One form of experimental mathematics involves using a computer to investigate mathematical objects and then using this data to form conjectures on their general properties, which are easier to prove *a posteriori*. Here, a mathematician may write a code to generate sufficiently many examples, study these examples, and identify (or even program the computer to guess) patterns among them.

For example, Chapter 1 implements this approach to study the reduced Gröbner bases of ideals generated by elementary symmetric functions. With a Maple package co-written with Doron Zeilberger, `Solomon.txt`, we can efficiently generate the reduced Gröbner bases of many *specific* ideals using symbolic computation. Through experimental methods, we deduce a pattern for the reduced Gröbner bases of the ideals $\langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle$ and $\langle e_{1,n}(x), e_{k,n}(x)\rangle$ for arbitrary $k \leq n$, and prove them by combinatorial means. We then find a basis for the general case $\langle e_{k_1,n}(x), \ldots, e_{k_m,n}(x)\rangle$, proving that it generates the ideal and showing empirically that it is a Gröbner basis. The work presented in this chapter was published in the Electronic Journal of Combinatorics [7].

The joint work with Robert Dougherty-Bliss presented in Chapter 2 also takes this approach, using Doron Zeilberger's Maple package `DyckClever.txt` from [14] to study the generating functions of restricted Dyck paths. Let A, B, C, and D be arbitrary sets of positive integers – either finite sets, infinite sets defined by arithmetical progressions, or the finite union of such sets. `DyckClever.txt` includes algorithms which directly compute the bivariate polynomial $F(t, X)$ such that $F(t, f(t)) = 0$, where $f(t)$ is the generating function for the sequence of the number of Dyck paths of semi-length n which avoid peak heights in A, valley heights in B, upward-runs with lengths in C, and downward-runs with lengths in D. While these algorithms can find the desired equations satisfied by the generating functions for given sets A, B, C, and D, they cannot produce identities for infinite families, or arbitrary sets of a given form. For example, running the appropriate procedure outputs the desired equation for the sequence of Dyck paths avoiding upward-runs of length $ar + b$ for *given* non-negative integers a and b (e.g. $5r + 2$), where r is a variable ranging over the non-negative integers. By running the procedure for various values a and b, we extend these findings to the infinite family of Dyck paths avoiding upward-runs of length $ar + b$ for *arbitrary* non-negative integers a and b. Using this approach, we form conjectures on other infinite families of restricted Dyck paths and prove that certain infinite families have an explicit context-free grammar which yields the equation satisfied by the generation function. The work in this chapter was published in Integers [8].

In another form of experimental mathematics, if the mathematician already has a conjecture, they can program the computer to either prove or disprove it. For example, say the mathematician can prove that, if a counterexample exists, then the minimal counterexample must have a given form. The computer can then either show that such a minimal counterexample cannot exist – which is famously done in the Appel-Haken proof of the Four Color Theorem [1] – or find a counterexample.

Furthermore, the mathematician may program the computer to automatically form and prove a conjecture. In the code, the mathematician provides the general form of the desired solution and outline of the proof. The computer can then try to identify patterns and form rigorous proofs by following the outline provided by the mathematician. For example, the mathematician may translate certain properties of a mathematical object into a system of polynomial equations. Such polynomials generate an ideal. Since any basis will give the same set of solutions, it can be advantageous to change bases. To form and prove conjectures, the computer may for instance use Buchberger's algorithm to compute a Gröbner basis, making it easier to manipulate and solve the system of equations.

We use this approach in Chapter 3, where we generalize Zeilberger's method for the automatic counting of restricted Dyck paths in [14] to the Motzkin paths. This chapter describes the procedures implemented in the accompanying Maple packages

Motzkin.txt and MotzkinClever.txt, which include programs to find the polynomial $F(t, X)$ such that $F(t, f(t)) = 0$, where $f(t)$ the generating function for the sequence counting Motzkin paths of length n avoiding peak heights in A, valley heights in B, upward-runs with lengths in C, downward-runs with lengths in D, and flat-runs with lengths in E. Motzkin.txt uses numeric dynamic programming to generate sufficiently many terms of the sequence enumerating Motzkin paths with the desired restrictions, and then guesses the recurrence. MotzkinClever.txt generates a finite system of algebraic equations by using symbolic dynamic programming and then solves the system to get the algebraic equation satisfied by the generating function directly. The work presented in this chapter was published in Enumerative Combinatorics and Applications [5].

This experimental approach is also demonstrated in Chapter 4 and Chapter 5, where we study generalized Dyck paths. Generalized Dyck paths are paths in the $xy-$plane from the origin $(0, 0)$ to $(n, 0)$ with an arbitrary set of atomic steps and that never go below the $x-$axis. Chapter 4 covers joint work with Doron Zeilberger, where we use symbolic dynamic programming to automatically generate $F(t, X)$ such that $F(t, X) = 0$ when $X := f(t)$, the weight enumerator for such paths of length n. These methods are fully automated in the Maple package GDW.txt. This work has been uploaded to the arXiv [9] and has been submitted for publication.

In Chapter 5, we begin to study the area under generalized Dyck paths. We continue to discuss the joint work with Doron Zeilberger found in [9]. We present similar methods as those presented in Chapter 4 to generate the algebraic equation satisfied by the generating function for the total area under generalized Dyck paths of length n. These methods are also fully automated in the accompanying Maple package GDW.txt. The rest of the chapter contains research from the article [6], which is uploaded to the arXiv and has been submitted for publication. We describe how to find the weight enumerator for such paths when, instead of a set of steps S, we are given bivariate polynomials $P(t,q)$, $Q(t,q)$, and $R(t,q)$ such that the weight enumerator $f(t,q)$ satisfies

$$f(t,q) = P(t,q) + Q(t,q)f(t,q) + R(t,q)f(t,q)f(qt,q).$$

We then present a method for finding $f^{(k)}(x,1) := \dfrac{d^k}{dq^k}[f(t,q)]\Big|_{q=1}$. Rather than outputting the algebraic equations presented earlier, this procedure produces closed-form expressions in terms of radicals. We demonstrate these methods with the bivariate weight enumerators for both Motzkin paths and Dyck paths with length n and area m. Moreover, we show how these procedures can be used to produce the Maclaurin series of $\frac{d^k}{dq^k}[f(t,q)]\Big|_{q=1}$, allowing us to find the generating function for the total area under such paths of length n as well as for the sum of a given power of the areas. We implemented these methods in the Maple package `qEW.txt`.

Gröbner Bases

Multiple procedures discussed in this dissertation implement Buchberger's algorithm and Gröbner bases to form and prove conjectures. Designing such algorithms that use Gröbner bases for efficient computations is a key problem in computer algebra. For potential readers unfamiliar with Gröbner bases, we will briefly elaborate on their general use in proofs.

Definition 0.1. A **Gröbner basis** of an ideal $I \subset k[x_1, ..., x_n]$ (with respect to a monomial order $>$) is a finite subset $G = \{g_1, ..., g_t\}$ of I such that, for every nonzero polynomial f in I, f is divisible by the leading term of g_i for some i.

Definition 0.2. A Gröbner basis G is **reduced** if, for every element $g \in G$,

1. the leading coefficient of g is 1, and
2. no monomial in g is in the ideal generated by the leading terms of the other elements in G.

It is known that every nonzero polynomial ideal I has a unique reduced Gröbner basis. In general, the Gröbner basis makes it easier to interpret the properties and structure of the ideal. It simplifies solving the ideal membership problem and finding solutions to a system of polynomial equations. A polynomial f lies in the ideal $I \subset k[x_1, ..., x_n]$ with Gröbner basis G if and only if the remainder on division of f by G is zero.

In forming conjectures, however, our problems will not already be stated as polynomials. By letting variables represent certain properties, we can translate various structures into polynomials, as was done in `MotzkinClever.txt`, the Maple package used in Chapter 3. Thus, if we let I be the ideal generated by the polynomials describing the properties of the given object, and f be the polynomial representing some claim about the mathematical object, then this claim is true if and only if f is in I. Therefore, the computer can prove or disprove the claim algorithmically, using Buchberger's algorithm to find a Gröbner basis and then applying the division algorithm. A more in depth explanation of Gröbner bases as well as examples can be found in [25] and [26].

Chapter 1

Gröbner Bases for Ideals Generated by Elementary Symmetric Functions

In their paper [21], Mora and Sala use computational and algebraic means to find the reduced Gröbner basis of the ideal generated by the elementary symmetric polynomials in n variables of degrees $d = 1, \ldots, n$. Haglund, Rhoades, and Shimonozo expand upon this, finding the reduced Gröbner basis of the ideal of elementary symmetric polynomials in n variables of degree d for $d = n - k + 1, \ldots, n$ for $k \leq n$ [17]. In this chapter, we further generalize their findings by using symbolic computation and experimentation to conjecture the reduced Gröbner basis for the ideal generated by the elementary symmetric polynomials in n variables of arbitrary degrees and prove that it is in fact a basis of the ideal.

Definition 1.1. Let k and n be natural numbers. The *elementary symmetric poly-*

nomial of degree k in n variables $x_1, \dots, x_n$ is

$$e_{k,n}(x) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \dots x_{i_k}.$$

Definition 1.2. The *homogeneous symmetric polynomial* of degree k in n variables $x_1, \dots, x_n$ is

$$h_{k,n}(x) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} x_{i_1} \dots x_{i_k}.$$

Given a set or multiset S with elements in $\{1, \dots, n\}$, define the weight of S to be

$$wt(S) = \prod_{s \in S} x_s^{m(s)},$$

where $m(s)$ is the multiplicity of s in S. For example,

$$wt(\{1,2,5\}) = x_1 x_2 x_5, \text{ and } wt(\{1,1,3,4\}) = x_1^2 x_3 x_4.$$

Then, $e_{k,n}(x)$ (respectively, $h_{k,n}(x)$) is the weight enumerator of the sets (respectively, multisets) with cardinality k whose elements are in $\{1, \dots, n\}$. Moreover, considering subsets of $\{1, \dots, n\}$ which do and do not contain n separately, we have the following recursive definition

$$e_{k,n}(x) = \begin{cases} 0, & \text{if } n < k \\ 1, & \text{if } k = 0 \\ e_{k,n-1}(x) + x_n e_{k-1,n-1}(x), & \text{otherwise.} \end{cases}$$

Similarly, when looking at multisets, we get

$$h_{k,n}(x) = \begin{cases} 0, & \text{if } n = 0 \text{ and } k > 0 \\ 1, & \text{if } k = 0 \\ h_{k,n-1}(x) + x_n h_{k-1,n}(x), & \text{otherwise.} \end{cases}$$

We use the recursive definitions to write Maple functions `eknS(x,k,n)` and `hknS(x,k,n)`, which output $e_{k,n}(x)$ and $h_{k,n}(x)$, respectively. These functions – along with others used to investigate the Gröbner basis of ideals generated by elementary symmetric polynomials – can be found in the accompanying Maple package `Solomon.txt`, written by AJ Bu and Doron Zeilberger.

In [21], Mora and Sala proved that $\{h_{1,n}(x), h_{2,n-1}(x), \ldots, h_{n,1}(x)\}$ is a Gröbner basis of the ideal $\langle e_{1,n}(x), \ldots, e_{n,n}(x)\rangle$. Using the accompanying package to efficiently generate the reduced Gröbner bases of many *specific* ideals, we can extend their findings. We first use experimental methods to deduce a pattern for the reduced Gröbner bases of the ideals $\langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle$ and $\langle e_{1,n}(x), e_{k,n}(x)\rangle$ for arbitrary $k \le n$, and prove them by combinatorial means. We then investigate other cases to expand upon our results to the ideal $\langle e_{k_1,n}(x), \ldots, e_{k_m,n}(x)\rangle$. We find a basis for this general case, proving that it generates the ideal, and show empirically that it is a Gröbner basis.

1.1 The Ideal $\langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle$

The procedure `Gkn(k,n,x)` in `Solomon.txt` outputs the reduced Gröbner basis (with respect to lexicographical order where $x_n > x_{n-1} > \cdots > x_1$) for the ideal $\langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle$. After running the procedure for multiple values k and n, we can conjecture that the reduced Gröbner basis is $\{h_{i,n-i+1}(x)|i = 1 \ldots k\}$. Indeed, for the case $k = n$, this agrees with the Gröbner basis that Mora and Sala proved in their paper [21]. In order to prove our conjecture, we use the following two relations between the elementary and homogeneous symmetric polynomials. This is essentially a well-known classical identity that can be found in [20], Eq. (2.6'). It has a very quick proof using generating functions, which is left to the reader. Nevertheless, we prefer the following somewhat longer, but more insightful combinatorial proof, inspired by Zeilberger's proof [30].

Lemma 1.1. *Let k and n be natural numbers. Then*

$$h_{k,n-k+1}(x) = \sum_{i=1}^{k} (-1)^{i+1} e_{i,n}(x) h_{k-i,n-k+1}(x)$$

Proof. This is equivalent to proving

$$\sum_{i=0}^{k} (-1)^{i} e_{i,n}(x) h_{k-i,n-k+1}(x) = 0.$$

This is trivial when $k > n$ because $h_{k-i,n-k+1}(x) = 0$ when $0 \leq i \leq k-1$, and $e_{k,n} = 0$. So, assume $k \leq n$. Then, the left-hand side is the weight enumerator of the set $\mathcal{S}_{k,n}$ of pairs (A, B), where

- A is a subset of $\{1, \ldots, n\}$ of order $|A|$,
- B is a multiset with cardinality $k - |A|$ whose elements are in $\{1, \ldots, n-k+1\}$,

and the weight of (A, B) is

$$w(A, B) = (-1)^{|A|} wt(A) wt(B).$$

Let $f : \mathcal{S}_{k,n} \to \mathcal{S}_{k,n}$ be defined as

$$f(A,B) = \begin{cases} (A \cup \{\min(B)\}, B - \{\min(B)\}), & \text{if } \min(B) < \min(A) \\ (A \smallsetminus \{\min(A)\}, B + \{\min(A)\}), & \text{otherwise.} \end{cases}$$

Note that this mapping is defined for all possible pairs of sequences, and it changes sign since the size of the first subset is either increasing or decreasing by 1. Moreover, if $\min(A) > \min(B)$ then

$$\begin{aligned} f(A,B) &= (A \cup \{\min(B)\}, B - \{\min(B)\}) =: (A', B'), \text{ and} \\ f(A', B') &= (A, B), \end{aligned}$$

since clearly $\min(A') = \min(B) \leq \min(B')$. If $\min(A) \leq \min(B)$ then

$$\begin{aligned} f(A,B) &= (A \smallsetminus \{\min(A)\}, B + \{\min(A)\}) =: (A', B'), \text{ and} \\ f(A', B') &= (A, B), \end{aligned}$$

since $\min(B') = \min(A) < \min(A')$. Thus, all elements of $\mathcal{S}_{k,n}$ can be paired up into mutually cancelling pairs, concluding our proof. $\square$

Lemma 1.2. *For any $n, k \in \mathbb{N}$,*

$$e_{k,n}(x) = \sum_{i=1}^{k} (-1)^{i+1} h_{i,n-i+1}(x) e_{k-i,n-i}(x).$$

Proof. Note that this is equivalent to

$$\sum_{i=0}^{k} (-1)^{i} h_{i,n-i+1}(x) e_{k-i,n-i}(x) = 0.$$

Again, this is trivial for $k > n$, so assume $k \leq n$. The left-hand side is the weight enumerator of the set $\mathcal{S}_{k,n}$ of ordered pairs (A, B), where

- A is a multiset with elements in $\{1, \ldots, n-|A|+1\}$, where $|A|$ is the cardinality of A,
- B is a subset of $\{1, \ldots, n-|A|\}$ of order $|B| := k-|A|$,

and the weight of (A, B) is

$$w(A, B) = (-1)^{|A|} wt(A) wt(B).$$

Let $f : \mathcal{S}_{k,n} \to \mathcal{S}_{k,n}$ be defined as

$$f(A, B) = \begin{cases} (A + \{\min(B)\}, B \smallsetminus \{\min(B)\}), & \text{if } \min(B) < \min(A) \\ (A - \{\min(A)\}, B \cup \{\min(A)\}), & \text{otherwise.} \end{cases}$$

As in the previous proof, this involution pairs all elements of $\mathcal{S}_{k,n}$ into mutually cancelling pairs. $\square$

We use the polynomial identities in the preceding lemmas to construct the Gröbner basis of the ideal $\langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle$ generated by the elementary symmetric polynomials of low degree. To start, we determine a basis of this ideal.

Lemma 1.3. *Let k and n be natural numbers such that $k \leq n$.*

$$\langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle = \langle h_{1,n}(x), h_{2,n-1}(x), \ldots, h_{k,n-k+1}(x)\rangle.$$

Proof. For $i = 1, \ldots, k$, we have $h_{i,n-i+1}(x) \in \langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle$, and $e_{i,n}(x) \in \langle h_{1,n}(x), h_{2,n-1}(x), \ldots, h_{k,n-k+1}(x)\rangle$ by Lemmas 1.1 and 1.2, respectively. It immediately follows that $\langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle = \langle h_{1,n}(x), h_{2,n-1}(x), \ldots, h_{k,n-k+1}(x)\rangle$. $\square$

Proposition 1.1. *Let k and n be natural numbers. The set $G := \{h_{i,n-i+1}(x) \mid 1 \leq i \leq k\}$ is the reduced Gröbner basis of the ideal $\langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle$ with respect to lexicographical order, where $x_n > x_{n-1} > \cdots > x_1$.*

Proof. By Lemma 1.3, the set G generates the ideal $I := \langle e_{1,n}(x), \ldots, e_{k,n}(x)\rangle$. The S-polynomial of any two distinct elements $h_{i,n-i+1}(x)$ and $h_{j,n-j+1}(x)$ in G is

$$\begin{aligned} S(h_{i,n-i+1}(x), h_{j,n-j+1}(x)) &= x_{n-j+1}^{j} h_{i,n-i+1}(x) - x_{n-i+1}^{i} h_{j,n-j+1}(x) \\ &= h_{j,n-j+1}(x) \sum_{\ell=0}^{i-1} x_{n-i+1}^{\ell} h_{i-\ell,n-i}(x) \\ &\quad - h_{i,n-i+1}(x) \sum_{\ell=0}^{j-1} x_{n-j+1}^{\ell} h_{j-\ell,n-j}(x). \end{aligned}$$

To prove the second equality, note that it is equivalent to

$$h_{i,n-i+1}(x) \sum_{\ell=0}^{j} x_{n-j+1}^{\ell} h_{j-\ell,n-j}(x) = h_{j,n-j+1}(x) \sum_{\ell=0}^{i} x_{n-i+1}^{\ell} h_{i-\ell,n-i}(x).$$

$x_{n-j+1}^{\ell} h_{j-\ell,n-j}(x)$ is the weight enumerator of all multisets of cardinality j with elements taken from $\{1, \ldots, n-j+1\}$, where $n-j+1$ appears exactly ℓ times. Thus, it is clear that

$$\sum_{\ell=0}^{j} x_{n-j+1}^{\ell} h_{j-\ell,n-j}(x) = h_{j,n-j+1}(x).$$

It follows that

$$\begin{aligned} S(h_{i,n-i+1}(x), h_{j,n-j+1}(x)) &= h_{j,n-j+1}(x) \sum_{\ell=0}^{i-1} x_{n-i+1}^{\ell} h_{i-\ell,n-i}(x) \\ &\quad - h_{i,n-i+1}(x) \sum_{\ell=0}^{j-1} x_{n-j+1}^{\ell} h_{j-\ell,n-j}(x). \end{aligned}$$

Moreover, for $i \neq j$

$$\begin{aligned} LT\left(h_{i,n-j+1}(x)\sum_{\ell=0}^{i-1} x_{n-i+1}^{\ell} h_{i-\ell,n-i}(x)\right) &= x_{n-j+1}^{j} x_{n-i+1}^{i-1} x_{n-i} \\ &\neq x_{n-i+1}^{i} x_{n-j+1}^{j-1} x_{n-j} \\ &= LT\left(h_{i,n-i+1}(x)\sum_{\ell=0}^{j-1} x_{n-j+1}^{\ell} h_{j-\ell,n-j}(x)\right). \end{aligned}$$

Hence,

$$LT(S(h_{i,n-i+1}(x), h_{j,n-j+1}(x))) = \max(x_{n-j+1}^{j} x_{n-i+1}^{i-1} x_{n-i}, x_{n-i+1}^{i} x_{n-j+1}^{j-1} x_{n-j})$$

and, by the division algorithm,

$$\overline{S(h_{i,n-i+1}(x), h_{j,n-j+1}(x))}^{G} = 0.$$

Therefore, G is a Gröbner basis of I by Buchberger's Criterion. Furthermore, G is a reduced Gröbner basis because, for any distinct i, j, $LT(h_{i,n-i+1}(x)) = x_{n-i+1}^{i}$ cannot divide the terms in $h_{j,n-j+1}(x)$. This follows from the fact that the terms of $h_{j,n-j+1}(x)$ have lower degree if $i > j$, and they cannot be multiples of x_{n-i+1} if $i < j$. $\square$

1.2 Investigation into the General Case

The procedure `GSn(S,n,x)` in `Solomon.txt` inputs a set $S = \{k_1, \ldots, k_m\}$, a non-negative integer n, and a variable x. It outputs the reduced Gröbner basis (with respect to lexicographical order where $x_n > x_{n-1} > \cdots > x_1$) for the ideal $\langle e_{k_1,n}(x), \ldots, e_{k_m,n}(x)\rangle$. Using this procedure to analyze the reduced Gröbner bases for various ideals, we conjecture the following basis for arbitrary S and n.

Proposition 1.2. *Let $k_1, \ldots, k_m$, and n be positive integers such that $1 \leq k_1 < \cdots < k_m \leq n$. Let I be the ideal*

$$I := \langle e_{k_1,n}(x), \ldots, e_{k_m,n}(x) \rangle,$$

and let M be the set of matrices of the form

$$\begin{bmatrix} e_{k_m-i_{m-1},n-i_{m-1}}(x) & \ldots & e_{k_m-i_1,n-i_1}(x) & e_{k_m,n}(x) \\ \vdots & \ldots & & \vdots \\ e_{k_1-i_{m-1},n-i_{m-1}}(x) & \ldots & e_{k_1-i_1,n-i_1}(x) & e_{k_1,n}(x) \end{bmatrix},$$

where $i_1 \in \{1, 2, \ldots, k_1, k_2\}$ and $i_j \in \{i_{j-1}+1, i_{j-1}+2, \ldots, k_j, k_{j+1}\}$ for $j > 1$. Then the set $G := \{det(m) \mid m \in M\}$ is a basis of I.

Proof. Note that the entries of the last column of any matrix in M are the elementary symmetric polynomials

$$e_{k_1,n}(x), \ldots, e_{k_m,n}(x)$$

that generate I. It immediately follows that $\langle G \rangle \subseteq I$.

For the other containment, let m_1 be the matrix in M where $i_j = k_{j+1}$. Then,

$$\det(m_1) = \det \left(\begin{bmatrix} 1 & e_{k_m-k_{m-1},n-k_{m-1}}(x) & \ldots & e_{k_m-k_2,n-k_2}(x) & e_{k_m,n}(x) \\ 0 & 1 & \ldots & e_{k_{m-1}-k_2,n-k_2}(x) & e_{k_{m-1},n}(x) \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & e_{k_2,n}(x) \\ 0 & 0 & \ldots & 0 & e_{k_1,n}(x) \end{bmatrix} \right)$$
$$= e_{k_1,n}(x).$$

Therefore, $e_{k_1,n}(x) \in \langle G\rangle$. Now suppose that for $L > 1$, $e_{k_\ell,n}(x) \in \langle G\rangle$ for all $1 \le \ell < L$. Let m_L denote the matrix in M such that $i_j = k_j$ for $j < L$ and $i_j = k_{j+1}$ for $j \ge L$. Then,

$$m_L = \begin{bmatrix} A_L & B_L \\ 0 & C_L \end{bmatrix},$$

where A is an $(m-L)\times(m-L)$ triangular matrix with whose diagonal entries are all 1, and 0 is an $L \times (m-L)$ zero matrix. Therefore, $\det m_L = \det C_L$, where C_L is the $L \times L$ matrix

$$\begin{bmatrix} e_{k_L-k_{L-1},n-k_{L-1}}(x) & e_{k_L-k_{L-2},n-k_{L-2}}(x) & \dots & e_{k_L-k_1,n-k_1}(x) & e_{k_L,n}(x) \\ 1 & e_{k_{L-1}-k_{L-2},n-k_{L-2}}(x) & \dots & e_{k_{L-1}-k_1,n-k_1}(x) & e_{k_{L-1},n}(x) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & e_{k_2-k_1,n-k_1}(x) & e_{k_2,n}(x) \\ 0 & \dots & 0 & 1 & e_{k_1,n}(x) \end{bmatrix}.$$

Define c_i to be the $(L-1)\times(L-1)$ matrix formed by removing the $i-th$ row and the last column from C_L. Then,

$$\det(C_L) = \sum_{i=1}^{L} (-1)^{i+1} e_{k_i,n}(x) \det(c_{L+1-i}).$$

Since $\det c_1 = 1$, it follows that

$$\begin{aligned} \det M &= \det C_L \\ &= (-1)^{L+1} e_{k_L,n}(x) + \sum_{i=1}^{L-1} (-1)^{i+1} e_{k_i,n}(x) \det(c_{L+1-i}). \end{aligned}$$

Since $\det M \in \langle G\rangle$ and, by our inductive hypothesis, $e_{k_i,n}(x) \in \langle G\rangle$ for $1 \le i \le L-1$, it follows $e_{k_L,n}(x)$ is in the ideal as well. Thus, $e_{k_i,n}(x) \in \langle G\rangle$ for $i = 1, \dots, m$, and $I = \langle G\rangle$. □

Since we found this basis by studying specifically the reduced Gröbner bases of various ideals, we further conjecture that it is the reduced Gröbner basis for arbitrary S and n. Indeed, the following proposition states that this conjecture holds for the ideal $\langle e_{1,n}(x), e_{k,n}(x)\rangle$.

Proposition 1.3. *Let k and n be natural numbers such that $n \geq k$. Let I be the ideal $\langle e_{1,n}(x), e_{k,n}(x)\rangle$, and let M be the set of matrices*

$$M = \left\{ \begin{bmatrix} 1 & e_{k,n}(x) \\ 0 & e_{1,n} \end{bmatrix}, \begin{bmatrix} e_{k-1,n-1}(x) & e_{k,n}(x) \\ 1 & e_{1,n}(x) \end{bmatrix} \right\}.$$

Then the set

$$G := \{det(m) \mid m \in M\}$$

is the reduced Gröbner basis of I with respect to lexicographical order, where $x_n > x_{n-1} > \cdots > x_1$.

Proof. By Proposition 1.2, G generates $I := \langle e_{1,n}(x), e_{k,n}(x)\rangle$. Note that by evaluating the determinants and then using the recursive properties of the elementary symmetric polynomials, we can rewrite G as

$$G = \{e_{1,n}(x), e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)\}.$$

Taking the S-polynomial of the elements in G, we have

$$\begin{aligned} S(e_{1,n}(x), &e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)) \\ &= x_{n-1}^2 x_{n-2} \dots x_{n-k+1} e_{1,n}(x) - x_n\big(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)\big) \\ &= \big(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)\big)e_{1,n-1}(x) \\ &\quad - e_{1,n}(x)\big(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x) - x_{n-1}^2 x_{n-2} \dots x_{n-k+1}\big) \end{aligned}$$

Note that the second equality obviously holds since it can be rewritten as

$$e_{1,n}(x)\big(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)\big) = \big(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)\big)e_{1,n}(x).$$

It is also clear that

$$\begin{aligned} &LT\big((e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x))e_{1,n-1}(x)\big) \\ &\qquad < LT\big(e_{1,n}(x)(e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x) - x_{n-1}^2 x_{n-2} \dots x_{n-k+1})\big), \end{aligned}$$

since the latter is a multiple of x_n. Hence,

$$\overline{S(e_{1,n}(x), e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x))}^G = 0,$$

and G is a Gröbner basis. It is clearly reduced since no term in $e_{1,n-1}(x)e_{k-1,n-1}(x) - e_{k,n-1}(x)$ is divisible by x_n and no term in $e_{1,n}(x)$ is divisible by x_{n-1}^2. □

Upon further investigation, we can show that our basis for the general case also gives the reduced Gröbner basis of the ideal $\langle e_{1,n}(x), \dots, e_{k,n}(x)\rangle$. We show that this is true in the following proposition, which is equivalent to Proposition 1.1.

Proposition 1.4. *Let k and n be positive integers such that $1 \leq k \leq n$. Let I be the ideal*

$$I := \langle e_{1,n}(x), \dots, e_{k,n}(x)\rangle,$$

and let M be the set of matrices of the form

$$\begin{bmatrix} e_{k-i_{k-1},n-i_{k-1}}(x) & \dots & e_{k-i_1,n-i_1}(x) & e_{k,n}(x) \\ \vdots & \dots & \vdots & \vdots \\ e_{1-i_{k-1},n-i_{k-1}}(x) & \dots & e_{1-i_1,n-i_1}(x) & e_{1,n}(x) \end{bmatrix},$$

where $1 \le i_1 < \cdots < i_{k-1} \le k$. *Then the set*

$$G := \{det(m) \mid m \in M\}$$

is the reduced Gröbner basis of I.

Proof. By Proposition 1.2, G is a basis of I. Moreover, by Proposition 1.1, the reduced Gröbner basis of I is $G' := \{h_{i,n-i}(x) \mid 1 \le i \le k\}$. Thus, it suffices to prove that $G = G'$.

For any positive integer L, let m_L denote the matrix such that no $i_j = L$. Then, as shown in the proof of Proposition 1.2,

$$\det m_L = \det C_{L,n},$$

where

$$C_{L,n} = \begin{bmatrix} e_{1,n-L+1}(x) & e_{2,n-L+2(x)} & \dots & e_{L-1,n-1}(x) & e_{L,n}(x) \\ 1 & e_{1,n-L+2}(x) & \dots & e_{L-2,n-1}(x) & e_{L-1,n}(x) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & e_{1,n-1}(x) & e_{2,n}(x) \\ 0 & \dots & 0 & 1 & e_{1,n}(x) \end{bmatrix}.$$

For any positive integers L and n where $L \le n$, we claim

$$\det C_{L,n} = h_{L,n-L+1}(x).$$

To prove this claim, we use induction over L, where for each $L \in \mathbb{N}$, we will show that the claim holds for all n. We begin with the base case; for any positive integer

n, we have $C_{1,n} = \begin{bmatrix} e_{1,n}(x) \end{bmatrix}$, so clearly

$$\begin{aligned} \det C_{1,n} &= e_{1,n}(x) \\ &= h_{1,n}(x). \end{aligned}$$

Now suppose that for any given $L \geq 2$, we have $\det C_{\ell,N} = h_{\ell,N-\ell+1}(x)$ for any $0 < \ell < L$ and $N > \ell$. Since

$$\det C_{L,n} = \sum_{i=1}^{L} (-1)^{i+1} e_{i,n}(x) \det(c_{L+1-i}),$$

where c_i is formed by removing the $i-th$ row and the last column from $C_{L,n}$, and we have shown that

$$h_{k,n-k+1}(x) = \sum_{i=1}^{k} (-1)^{i+1} e_{i,n}(x) h_{k-i,n-k+1}(x),$$

it is enough to show that $\det c_i = h_{i-1,n-L+1}$. Since c_1 is a triangular matrix whose diagonal entries are 1, it is obvious that

$$\begin{aligned} \det c_1 &= 1 \\ &= h_{0,n-L+1}. \end{aligned}$$

For $i > 1$, c_i can be written as

$$c_i = \begin{bmatrix} a_i & b_i \\ 0 & d_i, \end{bmatrix}$$

where d_i is an $(L-i) \times (L-i)$ triangular matrix whose diagonal entries are all 1, and $a_i = C_{i-1,n-L+i-1}$. Therefore,

$$\begin{aligned} \det c_i &= \det C_{i-1,n-L+i-1} \\ &= h_{i-1,n-L+1}, \end{aligned}$$

by our inductive hypothesis. Thus,

$$\begin{aligned} \det C_{L,n} &= \sum_{i=1}^{L} (-1)^{i+1} e_{i,n}(x) \det(c_{L+1-i}) \\ &= \sum_{i=1}^{L} (-1)^{i+1} e_{i,n}(x) h_{L-i,n-L+1}(x) \\ &= h_{L,n-L+1}(x), \end{aligned}$$

as desired. □

Further Study

One direction for further research is to formally prove that the basis we have found for the general case is the reduced Gröbner basis. We can also try to find similar identities for other ideals, such as those generated by various power sum symmetric polynomials or homogeneous symmetric polynomials of arbitrary degrees.

Chapter 2

Enumerating Restricted Dyck Paths with Context-Free Grammars

As Flajolet and Sedgewick masterfully demonstrate in their seminal text, *Analytic Combinatorics* [16], mathematicians have occasionally borrowed the study of formal languages from computer science and linguistics for combinatorial reasons. Many combinatorial classes can be reinterpreted as languages generated by certain grammars, and these grammars often make writing down generating functions, another favorite combinatorial tool, routine. Such grammars are sometimes called "combinatorial specifications."

For example, consider the well-known *Dyck paths.*

Definition 2.1. A Dyck path of semi-length n is a path in the $xy-$plane from the origin $(0, 0)$ to $(2n, 0)$ with atomic steps $U := (1, 1)$ and $D := (1, -1)$ that never goes below the $x-$axis.

Note that a Dyck path must have even length, and for this reason we often refer to Dyck paths of *semilength* n (length $2n$). The following are all Dyck paths:

$$UUDD$$

$$UDUD$$

$$UUUDUDDD$$

The number of Dyck paths of semilength n equals the nth *Catalan number*,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

There are many proofs of this fact, but here is a *grammatical* proof.

Let $\mathcal{P}$ denote the set of all Dyck paths. Then, $\mathcal{P}$ is generated by the unambiguous, context-free grammar

$$\mathcal{P} = \{EmptyPath\} \ \cup \ U\mathcal{P}D\mathcal{P}. \tag{2.1}$$

In words, a path is either empty or begins with a U, is followed by a Dyck path (shifted to height 1), a D, then another Dyck path.[1] This is a unique parsing of all Dyck paths.

Given a set of objects E each with a non-negative integer size, let $GF(E) = \sum_{k\geq 0} |E(k)|t^k$ be a formal generating function, where $|E(k)|$ is the number of objects of size k in E. The main result about formal grammars is that, in an unambiguous context free grammar,

$$GF(A \cup B) = GF(A) + GF(B),$$

[1] Note that D denotes the first time the path returns to height 0.

for disjoint clauses A and B, and

$$GF(AB) = GF(A)GF(B),$$

where $A \cup B$ is the union of the words of A and the words of B, and AB stands for "concatenation of words of A with words in B." The "sizes" of a grammar are the lengths of the words it generates.

In our case, if $f(t)$ is the generating function for the number of Dyck paths of semilength n, then the grammar (2.1) implies

$$\begin{aligned} f(t)) &= GF(\{EmptyPath\}) + GF(U\mathcal{P}D\mathcal{P}) \\ &= 1 + t\big[f(t)\big]^2. \end{aligned}$$

The generating function $C(t)$ for the Catalan numbers *also* satisfies

$$C(t) = 1 + t\big[C(t)\big]^2,$$

and since there are only two possible solutions, it is not hard to see that $f(t) = C(t)$.

The grammatical technique offers a unifying framework: Devise a grammar and you get an equation. Sometimes the equations turn out to be well-known or simple. Other times they are new and messy. The enumeration of all Dyck paths is one application of this framework, and here we want to demonstrate others. In particular, we will give grammatical proofs of several combinatorial facts about *restricted* Dyck paths, and also establish several infinite families of grammars in closed form.

First, let us define the restrictions we shall consider.

Definition 2.2. Given a Dyck path, the *height* of the path at position k is the partial sum of its first k terms. A *peak* of a Dyck path at height h (or simply "at h") is the bigram UD where the height of the path after the U is h. Similarly, a *valley* occurs at the bigram DU, and its height is analogously defined. The empty path has, by convention, a peak at 0 but no valley.

Definition 2.3. Given a sequence of steps L, define L^n to be the repetition of L n times. (For example, $U^2 = UU$ and $(UD)^3 = UDUDUD$.) A Dyck path has an *up-run of length* n provided that it contains at least one U^n that is not preceded nor followed by U. Similarly, it contains a *down-run of length* n provided that it contains at least one D^n that is neither preceded nor followed by D.

We will study Dyck paths whose peak heights, valley heights, up-run lengths, and down-run lengths avoid certain sets. We will, for example, discuss the set of all Dyck paths whose peak heights avoid $\{2, 4, 6, \dots\}$ and have no up-run of length greater than 2.

For arbitrary sets of positive integers A, B, C, and D, let $\mathcal{P}(A, B, C, D)$ be the set of Dyck paths whose peak heights avoid A, whose valley heights avoid B, whose up-run lengths avoid C, and whose down-run lengths avoid D. Let $f_{A,B,C,D}(t)$ be be the generating function for the number of Dyck paths of semi-length n in $\mathcal{P}(A, B, C, D)$.

Some of these sets have been studied. In [24], Peart and Woan provide a continued-fraction recurrence for the generating functions $f_{\{k\},\emptyset,\emptyset,\emptyset}(t)$. In [15], Eu, Liu, and Yeh take this idea further and express $f_{A,\emptyset,\emptyset,\emptyset}(t)$ as a finite continued fraction whenever A is finite or an arithmetic progression. In [18], Hein and Huang enumerate the number of Dyck paths which avoid up-runs of length k after a down step. In [14], Zeilberger presents a rigorous experimental method to derive equations for $f_{A,B,C,D}(t)$ when the sets involved are finite or arithmetic progressions. Proving "by hand" some of Zeilberger's interesting discoveries *ex post facto* was a motivation for the present work. We generalize some of Zeilberger's results to infinite families which are likely out of reach for symbolic methods.

Our results include several explicit grammars (and therefore generating function equations) for infinite families of the sets A and B, and also grammatical proofs of several interesting special cases suggested in [14]. Many of these—any grammars referencing restrictions on up- or down-runs—are not in [15]. Some of our results are suggested in the OEIS [22]; see, for example, A1006 (Motzkin numbers) and A004148 (generalized Catalan numbers).

The remainder of the chapter is organized as follows. Section 2.1 presents some results discovered by experimentation with software from [14] and proven with grammatical methods. Section 2.2 presents some infinite families of explicit grammars.

2.1 Combinatorial results

In this section we will present a number of results with grammatical proofs.

Proposition 2.1. *The number of Dyck paths of semilength n whose peak heights avoid $\{2r+3 \mid r \geq 0\}$ and whose up-runs are no longer than 2 is 1 when $n = 0$, and 2^{n-1} when $n \geq 1$.*

Proof. Let $\mathcal{P}$ be the set of all such Dyck paths, and $\mathcal{Q}$ the set of all Dyck paths which avoid peaks in $\{2r+2\}$ and up-runs longer than 2. Let their generating functions be $f(t)$ and $g(t)$, respectively. Note that $\mathcal{P}$ and $\mathcal{Q}$ satisfy the following grammar:

$$\mathcal{P} = \{EmptyPath\} \ \cup \ UD\mathcal{P} \ \cup \ UUD\mathcal{Q}D\mathcal{P}$$

$$Q = \{EmptyPath\} \ \cup \ UDQ.$$

This implies the following system of equations:

$$f(t) = 1 + tf(t) + t^2 g(t)f(t)$$
$$g(t) = 1 + tg(t).$$

Thus, $g(t) = (1-t)^{-1}$ (the only path in $\mathcal{Q}$ of semilength n is $(UD)^n$), and

$$f(t) = \frac{1-t}{1-2t}.$$

Therefore, $[t^0]f(t) = 1$, and $[t^n]f(t) = 2^{n-1}$. $\square$

The following proposition concerns *generalized Catalan numbers* (see A4148 in the OEIS and [28]). These numbers are defined by the recurrence

$$G_0 = 1$$
$$G_1 = 1$$
$$G_{n+2} = G_{n+1} + \sum_{1 \le k < n+1} G_k G_{n-k}.$$

Proposition 2.2. *The number of Dyck paths of semilength n whose peak heights avoid $\{2r+3 \mid r \geq 0\}$ and whose up-runs are no longer than 3 equals the $(n+1)$-th generalized Catalan number.*

Proof. Let $\mathcal{P}$, $\mathcal{O}$, and $\mathcal{E}$ be the set of all Dyck paths with up-runs no longer than 3, and whose peak heights avoid $\{2r+3 \mid r \geq 0\}$, $\{2r+2 \mid r \geq 0\}$, and $\{2r+1 \mid r \geq 0\}$, respectively. Let $f(t)$, $g(t)$, and $h(t)$ denote their generating functions, respectively. Observe that $\mathcal{P}$, $\mathcal{O}$, and $\mathcal{E}$ satisfy the following grammar:

$$\mathcal{P} = \{EmptyPath\} \cup\ UD\mathcal{P} \cup UUDO D\mathcal{P}$$

$$\mathcal{O} = \{EmptyPath\} \cup\ UD\mathcal{O} \cup UUUD\mathcal{O}D\mathcal{E}D\mathcal{O}$$

$$\mathcal{E} = \{EmptyPath\} \cup UUD\mathcal{O}D\mathcal{E}.$$

This grammar implies the following equations:

$$f(t) = 1 + tf(t) + t^2 g(t)f(t)$$

$$g(t) = 1 + tg(t) + t^3 h(t)\big[g(t)\big]^2$$

$$h(t) = 1 + t^2 g(t)h(t).$$

This system has two possible solutions for $f(t)$, but only one is holomorphic near the origin, namely

$$f(t) = \frac{2}{1 - t - t^2 + (t^4 - 2t^3 - t^2 - 2t + 1)^{1/2}}.$$

The generating function $G(t)$ for the generalized Catalan numbers is (see A4148 in the OEIS)

$$G(t) = \frac{1 - t + t^2 - (1 - 2t - t^2 - 2t^3 + t^4)^{1/2}}{2t^2},$$

and it is routine to verify that $G(t) = tf(t) + 1$. Therefore $G_{n+1} = [t^n]f(t)$ for $n \geq 0$. □

The following proposition is concerned with *Motzkin numbers* (see A1006 in the OEIS and [13]). A *Motzkin path* is like a Dyck path, but includes a “sideways” step S which does not change the height. The nth Motzkin number M_n is the number of Motzkin paths of length n. The generating function $M = M(t)$ for M_n satisfies the quadratic equation

$$M = 1 + tM + t^2 M^2.$$

There are numerous bijections between Motzkin paths and various restricted classes of Dyck paths. Such bijections are often variations of the "folding" map

$$\begin{aligned} UD &\mapsto S \\ DU &\mapsto S \\ UU &\mapsto U \\ DD &\mapsto D, \end{aligned}$$

which in general is not injective, but many restrictions on Dyck paths *make* it injective. For example, this idea shows that the Dyck paths of semilength n with no up-runs longer than 2 are in bijection with the Motzkin paths of length n. We offer a grammatical proof of this fact.

Proposition 2.3. *The number of Dyck paths of semilength n which avoid up-runs of length 3 or more equals the nth Motzkin number M_n.*

Proof. Let $\mathcal{P}$ be the set of such paths, and let $f(t)$ be the enumerator for paths in $\mathcal{P}$ of semi-length n. A grammar for $\mathcal{P}$ is

$$\mathcal{P} = \{EmptyPath\} \cup UUD\mathcal{P}D\mathcal{P} \cup UD\mathcal{P}.$$

Our grammar implies that

$$f(t) = 1 + tf(t) + t^2\big[f(t)\big]^2.$$

This is the same equation satisfied by the Motzkin generating function, and it is easy to check that $f(t) = M(t)$. $\square$

Proposition 2.4. *Consider the set of Dyck paths such that no peak or valley has* *positive, even height. The numbers of such paths of semilength $2n$ and $2n+1$ are $\binom{2n-1}{n}$ and $\binom{2n}{n}$, respectively.*

Proof. Let $\mathcal{P}$ denote the set of such paths, and let $\mathcal{O}$ denote the set of all Dyck paths whose peaks and valleys avoid odd heights. Let their weight enumerators be $f(t)$ and $g(t)$, respectively. These sets satisfy the following grammars

$$\mathcal{P} = \{EmptyPath\} \cup U\mathcal{O}D\mathcal{P},$$
$$\mathcal{O} = \{EmptyPath\} \cup UU\mathcal{O}DD\mathcal{O}.$$

This grammar can be translated into the following equations:

$$f(t) = 1 + tg(t)f(t), \text{ and}$$
$$g(t) = 1 + t^2\big[g(t)\big]^2$$

Solving this system for $g(t)$, we get two solutions for $g(t)$, but only the following is holomorphic near the origin

$$g(t) = \frac{1-\sqrt{1-4t^2}}{2t^2}.$$

Thus,

$$f(t) = \frac{2t-1-\sqrt{1-4t^2}}{2(2t-1)},$$

and it is easy to check that

$$[t^{2n}]f(t) = \binom{2n-1}{n}, \text{ and}$$
$$[t^{2n+1}]f(t) = \binom{2n}{n}.$$

□

Now, let us define a mapping which allows us to translate restrictions on up-run (respectively, down-run) lengths into restrictions on down-run (respectively, up-run) lengths. Let $\mathcal{P}$ denote the set of all Dyck paths. Define the mapping

$$\phi : \mathcal{P} \to \mathcal{P}, \qquad P \mapsto Q, \tag{2.2}$$

where applying ϕ reverses the order and direction of the steps in P. For example,

$$\phi(UUUDUUDDDDUD) = UDUUUUDDUDDD.$$

It is obvious that $\phi(P)$ must be a Dyck path. Moreover, it is easy to check that ϕ is an involution. Note that the up-runs (respectively, down-runs) in P become down-runs (respectively, up-runs) in $\phi(P)$ of the same length.

Proposition 2.5. *Let A and B be arbitrary sets of positive integers. The number of Dyck paths of semi-length n which avoid up-runs and down-runs with lengths in A and B, respectively, equals the number of Dyck paths of semi-length n which avoid down-runs with lengths in A and up-runs with lengths in B.*

Proof. Let $\mathcal{P}(A, B)$ be the set of Dyck paths such that no up-run has length in A and no down-run has length in B, and $\mathcal{P}(B, A)$ be the set of Dyck paths such that no up-run has length in B and no down-run has length in A. Then ϕ – defined in equation

2.2 – gives a one-to-one correspondence between the Dyck paths of semi-length n in $\mathcal{P}(A, B)$ and the Dyck paths of semi-length n in $\mathcal{P}(B, A)$. $\square$

Note that ϕ also allows us to translate the grammar of $\mathcal{P}(A, B)$ into the grammar of $\mathcal{P}(B, A)$, as seen in the following section.

2.2 Grammatical families

In this section we provide some explicit grammars for infinite families of restricted Dyck paths. In many cases, such grammars are guaranteed to exist. The reasoning in [14] shows that, for every set of Dyck paths whose peaks, valleys, and up- and down-runs avoid specific arithmetic progressions, we may construct a finite, context-free grammar which generates them. The method implied in [14] to compute these grammars gives no hint as to their *form*, and this is what we try to provide here.

Our first two results are about Dyck paths whose up-run lengths avoid a fixed arithmetic progression $\{Ar + B \mid r \geq 0\}$; each of these is accompanied by a corollary on Dyck paths that avoid down-run lengths in $\{Ar + B \mid r \geq 0\}$. It turns out that when $B < A$, there is a simple context-free grammar for such paths. When $B \geq A$ the situation is more complicated, but we can derive a “grammatical equation” which again leads to a generating function.

Proposition 2.6. *Let $B < A$ be non-negative integers. The set $\mathcal{P}$ of Dyck paths whose up-run lengths avoid $\{Ar + B \mid r \geq 0\}$ has the unambiguous grammar*

$$\mathcal{P} = \left(\bigcup_{\substack{0 \leq k < A \\ k \neq B}} U^k (D\mathcal{P})^k \right) \cup U^A (\mathcal{P}D)^A \mathcal{P},$$

and therefore

$$f(t) = \left(\sum_{\substack{0 \le k < A \\ k \neq B}} t^k \big[f(t)\big]^k \right) + t^A \big[f(t)\big]^{A+1},$$

where $f(t)$ is the weight-enumerator of $\mathcal{P}$.

Proof. The grammar clearly uniquely parses the empty path, so suppose that $P \in \mathcal{P}$ has length $n > 0$. Then P starts with a up-run of length $k > 0$ for some $k \not\equiv B$ mod A. If $k < A$, then write $P = U^k DW$, where W is a walk from height $k-1$ to height 0 with the same restrictions on up-runs as P. For $0 \le i < k-1$, let D_i indicate the down-step in W which hits the height i for the first time. Then

$$W = P_{k-1} D_{k-2} P_{k-2} D_{k-3} \ldots P_1 D_0 P_0,$$

where P_i is a Dyck path shifted to height i with the same restrictions on up-runs as P. This uniquely parses P into the case $U^k(D\mathcal{P})^k$ in the grammar.

If the initial up-run has length $k \ge A$, then write $P = U^A W$, where W is a walk from height A to height 0 whose up-run lengths avoid $\{Ar + B \mid r \ge 0\}$. By argument analogous to the previous paragraph, we can decompose W as

$$W = P_A D_{A-1} P_{A-1} D_{A-2} \ldots P_1 D_0 P_0,$$

where $P_i \in \mathcal{P}$. Thus W is of the form $(\mathcal{P}D)^A \mathcal{P}$, and this uniquely parses P into the final case of the grammar.

We have shown that $\mathcal{P}$ is contained in the language generated by this grammar, and it is easy to see that the first k cases of the grammar are contained in $\mathcal{P}$. The final case, $U^A(\mathcal{P}D)^A\mathcal{P}$, is also contained in the grammar, because concatenating U^A

to the beginning of a path does not change the length any of the up-runs modulo A. The different cases are clearly disjoint, so the grammar is also unambiguous. $\square$

Corollary 2.1. *Let $A, B \in \mathbb{Z}_{\geq 0}$ such that $B < A$. The set $\mathcal{P}$ of Dyck paths avoiding down-run lengths in $\{Ar + B | r \in \mathbb{Z}_{\geq} 0\}$ has the unambiguous grammar*

$$\mathcal{P} = \left(\bigcup_{\substack{0 \leq k < A \\ k \neq B}} (\mathcal{P}U)^k D^k \right) \cup \mathcal{P}(U\mathcal{P})^A D^A,$$

and therefore

$$f(t) = \left(\sum_{\substack{0 \leq k < A \\ k \neq B}} t^k \big[f(t)\big]^k \right) + t^A \big[f(t)\big]^{A+1},$$

where $f(t)$ is the weight-enumerator of $\mathcal{P}$.

Proof. Let ϕ be the involution defined in equation 2.2, and let $\mathcal{Q}$ be the set of Dyck paths avoiding up-run lengths in $\{Ar + B | r \in \mathbb{Z}_{\geq} 0\}$. By proposition 2.6,

$$\mathcal{Q} = \bigcup_{\substack{0 \leq k < A \\ k \neq B}} U^k (D\mathcal{Q})^k \cup U^A (\mathcal{Q}D)^A \mathcal{Q}.$$

Since

$$\begin{aligned} \phi(\mathcal{Q}) &= \mathcal{P}, \\ \phi(U^k (D\mathcal{Q})^k) &= (\mathcal{P}U)^k D^k, \text{ for all } 0 \leq k < A, \text{ and} \\ \phi(U^A (\mathcal{Q}D)^A \mathcal{Q}) &= \mathcal{P}(U\mathcal{P})^A U^A, \end{aligned}$$

ϕ translates the grammar of $\mathcal{Q}$ into the desired grammar for $\mathcal{P}$. $\square$

Proposition 2.7. *Let $A \leq B$ be nonnegative integers. The set $\mathcal{P}$ of Dyck paths* *avoiding up-run lengths in* $\{Ar + B \mid r \geq 0\}$ *satisfies the "grammatical equation"*

$$\mathcal{P} \cup U^B(D\mathcal{P})^B = \left(\bigcup_{0 \leq k < A} U^k(D\mathcal{P})^k\right) \cup \; U^A(\mathcal{P}D)^A\mathcal{P},$$

and therefore

$$f(t) + t^B\big[f(t)\big]^B = \left(\sum_{0 \leq k < A} t^k\big[f(t)\big]^k\right) + t^A\big[f(t)\big]^{A+1},$$

where $f(t)$ *is the weight-enumerator of* $\mathcal{P}$*.*

Note that the right-hand side is nearly identical to proposition 6; the difference being that we can get paths in $U^B(D\mathcal{P})^B$, which we will show below.

Proof. If P is a path in $\mathcal{P}$, then we can uniquely parse P into a case of the right-hand side by the same argument given in the previous proposition. Note that

$$\begin{aligned} U^B(D\mathcal{P})^B &= U^A U^{B-A}(D\mathcal{P})^B \\ &= U^A\{U^{B-A}(D\mathcal{P})^{B-A}\}(D\mathcal{P})^A \\ &= U^A[\{U^{B-A}(D\mathcal{P})^{B-A}\}D(\mathcal{P}D)^{A-1}]\mathcal{P}. \end{aligned}$$

The expression in brackets, $U^{B-A}(D\mathcal{P})^{B-A}$, is in $\mathcal{P}$, which shows that $U^B(D\mathcal{P})^B$ is contained in $U^A(\mathcal{P}D)^A\mathcal{P}$.

Conversely, it remains to show that the left-hand side is *all* that the right-hand side can generate. $\bigcup_{0 \leq k < A} U^k(D\mathcal{P})^k$ is contained in $\mathcal{P}$ as in the previous proposition. For $W \in U^A(\mathcal{P}D)^A\mathcal{P}$, write

$$W = U^A P_1 D \ldots P_A D P_{A+1}.$$

Let ℓ be the length of the initial up-run in P_1. If $\ell \not\equiv B \pmod{A}$, then W contains no up-runs of lengths in $\{Ar + B \mid r \geq 0\}$ and is a path in $\mathcal{P}$. If $\ell \equiv B \pmod{A}$, then $\ell \leq B - A$. If $\ell < B - A$ then the initial run of W has length less than B. Thus, W contains no up-runs of lengths in $\{Ar + B \mid r \geq 0\}$. For $\ell = B - A$, let D_i denote the first time W steps down to height i for $A < i < B$ and write

$$\begin{aligned} W &= U^A P_1 D \ldots P_A D P_{A+1} \\ &= U^A (U^{B-A} D_{B-1} W_{B-1} \ldots D_A W_A) D P_2 D \ldots P_A D P_{A+1} \\ &= U^B D_{B-1} W_{B-1} \ldots D_A W_A D P_2 D \ldots P_A D P_{A+1}. \end{aligned}$$

W_i is Dyck path shifted to height i by the definition of D_i. Hence, $W \in U^B(D\mathcal{P})^B$. □

Corollary 2.2. *Let $A, B \in \mathbb{Z}_{\geq 0}$ such that $B \geq A$. The set $\mathcal{P}$ of Dyck paths avoiding down-run lengths in $\{Ar + B | r \in \mathbb{Z}_{\geq} 0\}$ satisfies the grammatical equation*

$$\mathcal{P} \cup (\mathcal{P}U)^B D^B = \left(\bigcup\nolimits_{0 \leq k < A} (\mathcal{P}U)^k D^k\right) \cup \mathcal{P}(U\mathcal{P})^A D^A.$$

and therefore

$$f(t) + t^B \big[f(t)\big]^B = \left(\sum_{0 \leq k < A} t^k \big[f(t)\big]^k\right) + t^A \big[f(t)\big]^{A+1},$$

where $f(t)$ is the weight-enumerator of $\mathcal{P}$.

Proof. Let ϕ be the involution defined in equation 2.2, and let $\mathcal{Q}$ be the set of Dyck paths avoiding up-run lengths in $\{Ar + B | r \in \mathbb{Z}_{\geq} 0\}$. Applying ϕ to each clause of the grammar of $\mathcal{Q}$ given in proposition 2.7, we get

$$\mathcal{P} \cup (\mathcal{P}U)^B D^B = \left(\bigcup\nolimits_{0 \leq k < A} (\mathcal{P}U)^k D^k\right) \cup \mathcal{P}(U\mathcal{P})^A D^A,$$

as desired. □

Proposition 2.8. *Let $r \in \mathbb{Z}^+$. The set $\mathcal{P}$ of Dyck paths avoiding ascending and descending runs of lengths in $\{1, ..., r\}$ satisfies the grammatical equation*

$$\mathcal{P} \cup UD\mathcal{P} = \{EmptyPath\} \cup U^{r+1}D^{r+1}\mathcal{P} \cup U\mathcal{P}D\mathcal{P}.$$

and therefore

$$f(t) + tf(t) = 1 + t^{r+1}f(t) + t\big[f(t)\big]^2,$$

where $f(t)$ is the weight-enumerator of $\mathcal{P}$.

Proof. If $P \in \mathcal{P}$ is the empty path, then the grammar uniquely parses P. Otherwise, $P \in \mathcal{P}$ must begin with an ascending run of length $\ell > r$. If $\ell = r + 1$, then clearly U^{r+1} must be immediately followed by the descending run D^{r+1}, and P is uniquely parsed into the case $U^{r+1}D^{r+1}\mathcal{P}$.

If $\ell > r + 1$, then let D_0 denote the step where P returns to height 0 for the first time and write

$$P = UP_1D_0P_2.$$

It is obvious that $P_2 \in \mathcal{P}$ and P_1 is a Dyck path shifted to height 1. By restrictions on P, the final descending run in P_1 must have length $L \geq r$. If $L = r$ then the preceding ascending run ends at height $r+1$. But the ascending runs in P must have length of at least $r+1$, and hence P_1 hits height 0, contradicting the definition of D_0. From here, it is clear that P_1 has the same restrictions on ascending and descending runs as P. Thus, P is uniquely parsed into the case $U\mathcal{P}D\mathcal{P}$.

Since it is trivial that $UD\mathcal{P}$ is contained in $U\mathcal{P}D\mathcal{P}$, we have shown that the left-hand side of the given equation is generated by the right-hand side. It is also obvious that the cases defined on the right-hand side are disjoint and that $\epsilon \cup U^{r+1}D^{r+1}\mathcal{P}$ is contained in $\mathcal{P}$. A path $UP_1DP_2 \in U\mathcal{P}D\mathcal{P}$ is contained in $UD\mathcal{P}$ if P_1 is the empty path and $\mathcal{P}$ otherwise. Thus, $\mathcal{P}$ satisfies the given grammatical equation □

Proposition 2.9. *Let $m, n \in \mathbb{Z}^+$. The set $\mathcal{P}$ of Dyck paths avoiding ascending runs of lengths in $\{1, ..., m\}$ and descending runs of lengths in $\{1, ..., n\}$ satisfies the grammatical equation*

$$\mathcal{P} \cup UD\mathcal{P} = \{EmptyPath\} \cup U\mathcal{P}D\mathcal{P} \cup U^{m+1}D^{n+1}(\mathcal{P}D)^{m-n}\mathcal{P}, \text{ if } m \geq n \tag{2.3}$$

$$\mathcal{P} \cup \mathcal{P}UD = \{EmptyPath\} \cup \mathcal{P}U\mathcal{P}D \cup \mathcal{P}(U\mathcal{P})^{n-m}U^{m+1}D^{n+1}, \text{ if } m \leq n. \tag{2.4}$$

Proof. We have already shown that this statement is true for $m = n$. Suppose $m > n$. If $P \in \mathcal{P}$ is the empty path, then the grammar uniquely parses P. Otherwise, P must begin with an ascending run of length $\ell > m$. If $\ell = m + 1$ then U^{m+1} is followed by a descending chain of length of at least $n + 1$. Let D_i denote the first time P returns to height i for $0 \leq i \leq m - n - 1$, and write

$$P = U^{m+1}D^{n+1}P_{m-n}D_{m-n-1}...P_1D_0P_0.$$

It is obvious that P_i is a Dyck path, shifted to height i, that has the same restrictions on ascending runs and descending runs (with the exception of the final descending run) as P. Since P_i is a Dyck path, its final descending run must be at least as long as the ascending run preceding it. Thus, P_i is either the empty path or ends with a descending run of length $L > m > n$. Thus, P is uniquely parsed into the case $U^{m+1}D^{n+1}(\mathcal{P}D)^{m-n}\mathcal{P}$.

If $\ell > m+1$ then, letting D_0 denote the first time P returns to height 0, write

$$P = UP_1D_0P_0.$$

Clearly, $P_0 \in \mathcal{P}$, and P_1 is a Dyck path shifted to height 1 and has the same restrictions on ascending runs as P. Using the same argument as for P_i in the previous case, the descending runs in P_1 also have the same restrictions as P. This uniquely parses P into the case $U\mathcal{P}D\mathcal{P}$. Finally, it is obvious that $UD\mathcal{P}$ is contained in $U\mathcal{P}D\mathcal{P}$, so the left-hand side of (1) is generated by the right-hand side.

It is clear that the cases on the right-hand side are disjoint, and the empty path is an element of $\mathcal{P}$. Also, $UP_1DP_2 \in U\mathcal{P}D\mathcal{P}$ is contained in $\mathcal{P}$ if P_1 is not the empty path, and is contained in $UD\mathcal{P}$ otherwise. $U^{m+1}D^{n+1}(\mathcal{P}D)^{m-n}\mathcal{P}$ is contained in $\mathcal{P}$, since all ascending runs clearly avoid restrictions on $\mathcal{P}$ and the descending runs are formed by concatenating down-steps to descending runs of length of at least $n-1$. Thus, we have proved the grammar for the case $m \geq n$.

Now assume that $n \geq m$. Applying the involution ϕ from equation 2.2, we can directly translate the grammar 2.3 into the desired grammar 2.4. $\square$

Proposition 2.10. *Let $r, k \in \mathbb{Z}^+$ and let $\mathcal{P}$ be the set of Dyck paths avoiding ascending runs of length $\{1, ..., r\}$ and descending runs of length $\{k+1, ..., r\}$. Then the 'grammar' of $\mathcal{P}$ is*

$$\mathcal{P} \cup UD\mathcal{P} \cup U^{r+1}D^k(D\mathcal{P})^{r+1-k} = \{EmptyPath\} \cup U\mathcal{P}D\mathcal{P} \cup U^{r+1}D^{r+1}\mathcal{P} \cup U^{r+1}(DP)^{r+1}$$

Proof. If $P \in \mathcal{P}$ is the empty path, then the grammar uniquely parses P. Otherwise, P begins an ascending run of length $\ell > r$, and we can deduce that it also ends with a descending run of length $L > r$. If $\ell > r+1$, then let D_0 denote the first time that

P returns to the $x-$axis and write

$$P = UP_1D_0P_0.$$

It is easy to see that P_0 is a path in $\mathcal{P}$ and P_1 is a Dyck path shifted to height 1. The initial ascending run in P_1 has length $\ell - 1 > r$. Thus, all ascending runs in P_1 have length of at least $r+1$ and, since P_1 is a shifted Dyck path, the final descending run in P_1 must also have length of at least $r+1$. From here, it is easy to see that P_1 has the same restrictions on ascending and descending runs as P. P is therefore uniquely parsed into the case $U\mathcal{P}D\mathcal{P}$.

Suppose $\ell = r+1$. Let D_i be the step where P returns to height i for the first time and write

$$P = U^{r+1}D_rP_r...D_0P_0.$$

P_i is a Dyck path for all i and, if P_i is not the empty path, it must end with a descending run of length $r+1$ by restrictions on ascending runs. Thus P_i is a path in $\mathcal{P}$, and P is parsed into the case $U^{r+1}(D\mathcal{P})^{r+1}$.

It is trivial that $UD\mathcal{P}$ is contained in $U\mathcal{P}D\mathcal{P}$ and $U^{r+1}D^k(D\mathcal{P})^{r+1-k}$ is contained in $U^{r+1}(D\mathcal{P})^{r+1}$. Thus, the left-hand side is generated by the right-hand side. Note that, on the left-hand side,

$$UD\mathcal{P} \cap \mathcal{P} = UD\mathcal{P} \cap U^{r+1}D^k(D\mathcal{P})^{r+1-k} = \emptyset,$$

however

$$\mathcal{P} \cap U^{r+1}D^k(D\mathcal{P})^{r+1-k} = U^{r+1}D^{r+1}\mathcal{P}.$$

Looking at the right-hand side, it is clear that $\epsilon, U\mathcal{P}D\mathcal{P}$, and $U^{r+1}(D\mathcal{P})^{r+1}$ are disjoint, and $U^{r+1}D^{r+1}\mathcal{P}$ is contained in $U^{r+1}(D\mathcal{P})^{r+1}$. Note that this resolves the issue of double counting paths in $U^{r+1}D^{r+1}\mathcal{P}$ on the left-hand side. Thus, all that remains to show is that all the paths generated by the right-hand side are contained in the left-hand side.

The path UP_1DP_0 in $U\mathcal{P}D\mathcal{P}$ is clearly in $\mathcal{P}$ if P_1 is not the empty path and in $UD\mathcal{P}$ otherwise. For W in $U^{r+1}(D\mathcal{P})^{r+1}$, write

$$W = U^{r+1}D_rP_r...D_1P_1D_0P_0.$$

Choose the smallest i such that P_{r-i} is not the empty path or, if no such i exists, set $i = r$. Then the first descending run in W has length $i+1$. If $i \geq k$ then W is an element of $U^{r+1}D^k(D\mathcal{P})^{r+1-k}$. Otherwise, we claim that W is a path in $\mathcal{P}$. It is clear that W is a Dyck path and we have seen that nonempty $P_j \in \mathcal{P}$ must end in a descending run of length of at least $r+1$. Thus, we only need to show that the first descending run in W follows the restrictions in $\mathcal{P}$. This is clearly true since $i < k$. Hence $W \in \mathcal{P}$, and $\mathcal{P}$ satisfies the grammatical equation as desired. $\square$

Conclusion

We have given several grammatical proofs of various combinatorial results about restricted Dyck paths and established some infinite families of grammars. Our methods work because we are able to derive *context-free grammars* describing certain restricted classes Dyck paths, namely when our restrictions involved sets of arithmetic progressions.

It is natural to ask if context-free grammars exist for other types of restrictions.

Parikh's theorem [23] states that the set of lengths of any context-free language is the union of finitely-many arithmetic progressions, so it seems likely that restrictions involving arithmetic progressions are *essentially* all that can be done. However, addressing this question in full is beyond our current scope.

Chapter 3

Automated Counting of Motzkin Paths

Doron Zeilberger introduced methods of counting restricted Dyck paths using numeric dynamic programming and symbolic dynamic programming in his paper "Automatic Counting of Restricted Dyck Paths via (Numeric and Symbolic) Dynamic Programming" [14]. Here, I generalize his findings to the Motzkin paths with the accompanying maple packages, which are Motzkin analogues to Zeilberger's maple packages in [14].

Definition 3.1. A *Motzkin path of length* n is a path in the $xy-$plane from the origin $(0,0)$ to $(n,0)$ with atomic steps $U := (1,1)$, $D := (1,-1)$, and $F := (1,0)$ that never goes below the x-axis.

For example, the following paths are Motzkin paths of length 6:

$$UUUDDD, UDUFDF, UUFFDD, FUDFUD, FFFFFF.$$

To count the number of Motzkin paths with a given set of restrictions, let $\mathcal{P}$ denote the set of such paths and consider the weight enumerator

$$f(t) := \sum_{P \in \mathcal{P}} t^{Length(P)}.$$

Note that this equals the ordinary generating function

$$\sum_{n=0}^{\infty} a(n) t^n,$$

of the sequence $\{a(n)\}_{n=0}^{\infty}$, counting the Motzkin paths of length n with the desired restrictions.

This paper presents two methods for finding the polynomial $F(t, X)$ that is zero when $X := f(t)$. For example, let $\mathcal{P}$ denote the set of all Motzkin paths. Note that $P \in \mathcal{P}$ either is the empty path, begins with the step F, or begins with the step U. If P begins with the step F, then we can write

$$P = FP_0,$$

and it is obvious that P_0 must also be a Motzkin path. If P begins with the step U, then let D_0 denote the first time P returns to the $x-$axis and write

$$P = UP_1D_0P_2.$$

It is easy to see that P_1 must be a Motzkin path shifted to height 1 and P_2 is also a Motzkin path. Note that, for the paths in $\mathcal{P}$, these decompositions are unambiguous. Moreover, given any Motzkin paths P_0, P_1, and P_2, it is clear that the empty path,

FP_0, and UP_1DP_2 are also Motzkin paths. $\mathcal{P}$ therefore has the *grammar*

$$\mathcal{P} = \{EmptyPath\} \cup F\mathcal{P} \cup U\mathcal{P}D\mathcal{P}.$$

Thus, setting X equal to the weight enumerator of $\mathcal{P}$, we get the recurrence

$$X = 1 + tX + t^2X^2.$$

There are a fair number of papers that discuss the enumeration of certain families of Motzkin paths – [15], [3], and [2] to name a few. Recall that Dyck paths are also a family of restricted Motzkin paths, as they are Motzkin paths with no flat steps. In "Automatic Counting of Restricted Dyck Paths via (Numeric and Symbolic) Dynamic Programming" [14], Zeilberger considers Dyck paths with restrictions on peak heights, valley heights, upward-runs, and down-ward runs. In this paper, we will look at similar restrictions. Due to the allowance of flat-steps in Motzkin paths, however, we reevaluate what peaks and valleys are. We also introduce restrictions on flat-runs.

Definition 3.2. A Motzkin path has a *flat-run of length* n if it contains a run F^n that is not directly followed by nor directly follows a flat-step.

The Maple Packages

This chapter covers procedures in the following Maple packages:

- `Motzkin.txt`: Uses numeric dynamic programming to generate sufficiently many terms of the sequence of Motzkin paths with the desired restrictions, and then guesses the recurrence to get the desired equation.

- `MotzkinClever.txt`: Generates a finite system of algebraic equations by using symbolic dynamic programming and then solves the system to get the equation satisfied by the generating function directly.

These packages and example input and output files can be found at `https://ajbu1.github.io/Papers/AutocountMotzkin/AutocountMotzkin.html`.

3.1 Numeric Dynamic Programming (`Motzkin.txt`)

Let us start by looking at the most basic case - finding the number of all Motzkin paths of length N. By definition, every Motzkin path must end with either a down-step or a flat-step. If a Motzkin path ends with a downwards-run on length r, then the preceding run is either an ascending-run or a flat-run that ends at height r. We introduce the following notation.

$$\begin{aligned}
u(m,n) = &\text{ the number of walks from } (0,0) \text{ to } (m,n) \text{ that never goes below the } x\text{-axis} \\
&\text{ and ends with an up-step.} \\
d(m,n) = &\text{ the number of walks from } (0,0) \text{ to } (m,n) \text{ that never goes below the } x\text{-axis} \\
&\text{ and ends with a down-step.} \\
f(m,n) = &\text{ the number of walks from } (0,0) \text{ to } (m,n) \text{ that never goes below the } x\text{-axis} \\
&\text{ and ends with a flat-step.}
\end{aligned}$$

These give us the following equalities:

$$d(m,n) = \sum_{r=1}^{m} u(m-r,n+r) + f(m-r,n+r),$$
$$f(m,n) = \sum_{r=1}^{m} u(m-r,n) + d(m-r,n), \text{ and}$$
$$u(m,n) = \sum_{r=1}^{m} f(m-r,n-r) + d(m-r,n-r),$$

with the initial conditions $f(0,0) = 0 = u(0,0)$ and $d(0,0) = 1$, and the boundary conditions $d(m,k) = u(m,k) = f(m,k) = 0$ for $k > m$.

`Motzkin.txt` implements these equations through the procedures `u(m,n)`, `d(m,n)`, and `f(m,n)`. Thus, to get the first $N+1$ terms of the sequence $\{a(n)\}_{n=0}^{\infty}$ where $a(n)$ is defined to be the number of Motzkin paths of length n, run

```
seq(d(m,0)+f(m,0),m=0..N)
```

For example,

```
seq(d(m,0)+f(m,0),m=0..10)
```

outputs

```
1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188.
```

3.1.1 Restricted Motzkin Paths

Let A, B, C, D and E be arbitrary sets of positive integers – either finite sets or infinite sets defined by the union of arithmetic progressions. We consider restricted Motzkin paths that avoid

- peak heights in A,
- valley heights in B,
- upward-runs with lengths in C,
- downward-runs with lengths in D, and
- flat-runs with lengths in E.

In coming up with an analogue to $u(m,n)$, $d(m,n)$, and $f(m,n)$, we notice that flat-runs complicate how we count paths with restrictions on peak heights and valley heights. For example, the path

$$UUFUDDD$$

avoids peaks with height 2 even though it contains an upward-run that ends at height 2. To address this, we need to define subcases for $u(m,n), d(m,n)$ and $f(m,n)$ as follows:

$$u_d(m,n) = \begin{cases} 0 & \text{if } n \in A \\ u(m,n) & \text{otherwise} \end{cases},$$

$$d_u(m,n) = \begin{cases} 0 & \text{if } n \in B \\ d(m,n) & \text{otherwise} \end{cases},$$

$$f_u(m,n) = \sum_{\substack{1 \le r \le m \\ r \notin E}} u(m-r,n) + d_u(m-r,n)), \text{ and}$$

$$f_d(m,n) = \sum_{\substack{1 \le r \le m \\ r \notin E}} u_d(m-r,n) + d(m-r,n).$$

When counting restricted paths from $(0,0)$ to (m,n) ending in a downward run of length r, the preceding run is either an upward-run or a flat-run. If it is preceded by an upward-run that ends at a height in A, then the path violates the restriction on peak heights. Thus, we only want to consider the paths counted by $u_d(m-r,n+r)$. Otherwise, it is preceded by a flat-run. If this flat-run is preceded by an upward-run ending at a height in A, then the path again has a forbidden peak height. Thus, we are interested in exactly the paths counted by $f_d(m-r,n+r)$. Similarly, when counting restricted paths from $(0,0)$ to (m,n) ending in an upward-run of length r, we only consider the paths counted by $d_u(m-r,n-r)$ and $f_u(m-r,n-r)$ to avoid forbidden valley heights. Note that our definitions of $f_u(m,n)$ and $f_d(m,n)$ ensure that the sub-path being counted does not end in a flat-run of length in E. We can use similar restrictions to ensure that our paths do not contain any forbidden run lengths.

We set

$$d(m,n) = \sum_{\substack{1\le r\le m\\ r\notin D}} u_d(m-r,n+r) + f_d(m-r,n+r),$$

$$f(m,n) = \sum_{\substack{1\le r\le m\\ r\notin E}} d(m-r,n) + u(m-r,n), \text{ and}$$

$$u(m,n) = \sum_{\substack{1\le r\le m\\ r\notin C}} d_u(m-r,n-r) + f_u(m-r,n-r).$$

These functions are implemented in `Motzkin.txt` and are used to get

`SeqABCDE(A,B,C,D,E,N)` and `SeqABCDEr(A,B,C,D,E,r,N)`,

which generate the terms $a(n)$ – the number of Motzkin paths of length n with the desired restrictions – for $0 \le n \le N$. `SeqABCDE(A,B,C,D,E,N)` is used when A, B, C, D

and E are finite sets of non-negative integers, and `SeqABCDEr(A,B,C,D,E,r,N)` is used when the sets are defined by linear equations.

For example,

```
SeqABCDE({},{},{1},{1},{1},11)
```

outputs

```
[1, 0, 1, 1, 2, 1, 5, 4, 12, 13, 34, 38],
```

and

```
SeqABCDEr({2*r+1},{2*r+1},{},{},{},r,11)
```

outputs

```
[1, 1, 1, 1, 2, 6, 16, 36, 73, 145, 301, 661].
```

The first output tells us, for example, that there are four Motzkin paths of length 7 avoiding upward, downward, and flat runs of length 1. We can verify that this is true by noting that such paths must either be all flat steps or a permutation of three consecutive flat-steps, two consecutive up-steps, and two consecutive down-steps. Since the up-steps must occur before the down-steps by the definition of Motzkin paths, the set of desired paths is

$$\{FFFFFFF, FFFUUDD, UUDDFFF, UUFFFDD\}.$$

The second output states that there are six Motzkin paths of length 5 avoiding peaks and valleys with odd heights. We can easily check that the set of such paths is

$$\{FFFFF, FUUDD, UFUDD, UUDDF, UUDFD, UUFDD\}.$$

Note that

```
SeqABCDEr({},{},{},{},{r+1},r,30).
```

outputs the number of Motzkin paths of length n avoiding flat-steps for $n = 0, \ldots, 30$. This outputs 0 when n is odd, and the terms for even n give us the list

```
1,1,2,5,14,42,132,429,1430,4862,16796,58786,208012,742900,2674440,9694845.
```

Inputting these terms into OEIS, we can easily verify that this is in fact the sequence of the number of Dyck paths of semi-length n.

3.1.2 Finding the Equation Satisfied by the Generating Function

The desired $F(t, X)$ is a polynomial, so there exist polynomials $q_0(t), \ldots, q_d(t)$ such that

$$F(t, X) = q_0(t) + q_1(t)X + \ldots + q_d(t)X^d.$$

$F(t, X)$ is zero when $X := f(t)$, the generating function of the desired sequence, thus $f(t)$ is algebraic. Therefore, $f(t)$ satisfies a linear differential equation with polynomial coefficients, and so there is a linear recurrence equation with polynomial coefficients for the terms $a(n)$ in our sequence. (For more details see [19], particularly Sections 6.2 and 7.2.) To get the desired polynomial, we borrow directly from Zeilberger's method of using undetermined coefficients to guess the recurrence used in `Dyck.txt` in [14].

3.2 Symbolic Dynamic Programming

`MotzkinClever.txt` uses symbolic dynamic programming to find $F(t, X)$. More specifically, the recurrence for the set of restricted Motzkin paths is expressed as a polynomial by assigning different variables to different sets of restrictions. In addition to our original set of restricted Motzkin paths, we look at the "children" of this set. These are sets of Motzkin paths with other restrictions such that any element of our original set can be written in some form concatenating certain steps with such paths. This process is described more concretely below. We then continue to look at the children of the new sets until no new children can be produced. We will see that this must happen eventually, yielding a finite system of polynomial equations that contains the same number of equations as variables. We use this system of equations to find the equation satisfied by the generating function.

3.2.1 Avoiding Peak and Valley Heights in Finite Sets:

Let A and B be two arbitrary finite sets of non-negative integers. We consider the ordinary generating function $f_{A,B}$ of the sequence of Motzkin paths avoiding

- peak-heights in A, and
- valley-heights in B.

First, note that the sequence of walks with only flat-steps with weight $t^{Length(P)}$ has the generating function $\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$. For convention, we say that a path has a peak at height 0 if and only if it is a flat run. Now, let $\mathcal{P}$ denote the set of Motzkin paths avoiding peak heights in A and valley heights in B, and let $\mathcal{F}$ denote the set of flat runs. Consider the following three cases:

Case 1: If $0 \in A$ then let $A_1 := A \backslash \{0\}$.

Let $\mathcal{P}_1$ be the set of Motzkin paths avoiding peak heights in A_1 and valley heights in B. Then it is clear $\mathcal{P}$ is the union of the disjoint sets $\mathcal{F}$ and $\mathcal{P}$, giving the following grammar

$$\mathcal{P} \cup \mathcal{F} = \mathcal{P}_1.$$

This gives us the following equation

$$f_{A,B}(t) = f_{A_1,B}(t) - \frac{1}{1-t}.$$

Case 2: If $0 \notin A$ and $0 \in B$ then let $A_1 := \{a-1|a \in A\}$ and $B_1 := \{b-1|b \in B\backslash\{0\}\}$.

Let $\mathcal{P}_1$ denote the set of Motzkin paths avoiding peak heights in A_1 and valley heights in B_1. Then any non-flat path P in $\mathcal{P}$ starts with either an up-step or a flat-run followed by an up-step, and ends with either a down-step or a down-step followed by a flat-run. Note that, since P avoids valleys with height 0, it can only return to the $x-$axis once. We can therefore write

$$P = F^{k_1} U P_1 D F^{k_2},$$

where k_1 and k_2 are non-negative integers, and P_1 is some path in $\mathcal{P}_1$. Thus, we get the grammar

$$\mathcal{P} = \mathcal{F} \cup \mathcal{F} U \mathcal{P}_1 D \mathcal{F},$$

which gives the following equation

$$f_{A,B}(t) = \frac{1}{1-t} + \frac{t^2}{(1-t)^2} f_{A_1,B_1}(t).$$

Case 3: If $0 \notin A$ and $0 \notin B$ then let $A_1 := \{a-1|a \in A\}$ and $B_1 := \{b-1|b \in B\}$.

Let $\mathcal{P}_1$ denote the set of Motzkin paths avoiding peak heights in A_1 and valley heights in B_1. Any non-flat path P in $\mathcal{P}$ starts with either an up-step or a flat-run followed by an up-step. Then, letting D denote the first time P returns to the $x-$axis, we can write

$$P = F^k U P_1 D P',$$

where k is a non-negative integer, P_1 is some path in $\mathcal{P}_1$, and P' some path in $\mathcal{P}$. We then have the grammar

$$\mathcal{P} = \mathcal{F} \cup \mathcal{F} U \mathcal{P}_1 D \mathcal{P}.$$

Hence,

$$f_{A,B}(t) = \frac{1}{1-t} + \frac{t^2}{1-t} f_{A,B}(t) f_{A_1,B_1}(t).$$

Thus, $\mathcal{P}$ has the child $\mathcal{P}_1$. We then apply this procedure to $\mathcal{P}_1$ and so on. Note that we will eventually remove all the elements of A and B and will therefore have finitely many "descendants" of our original set. Moreover, since we have an equation to find the children of each variable, we have as many equations as variables. Each equation has only two variables, except the last equation which has one, and the variables are raised to degree at most 1. Thus, we can eliminate every variable except the one representing our original $f_{A,B}$ from the first generated equation. This gives us the polynomial satisfied by the generating function of the Motzkin paths with the desired restrictions.

This procedure is implemented in `MotzkinClever.txt` by the procedure `fAB(A,B,t,X)`. For example, say we want the equation satisfied by the generating function of the sequence $\{a(n)\}_{n=0}^{\infty}$, where $a(n)$ is the number of Motzkin paths avoiding peak heights in $\{1,4\}$ and valley heights in $\{1,3\}$. Running

```
fAB({1,4},{1,3},t,X)
```

outputs the polynomial

$$
\begin{aligned}
&t^8 - 2t^7 + 5t^6 - 12t^5 + 29t^4 - 38t^3 + 25t^2 - 8t + 1 \\
&\quad + (t^6 - 16t^3 + 24t^2 - 12t + 2)(-1+t)^3 X \\
&\quad + (t^6 + 2t^5 - t^4 - 8t^3 + 12t^2 - 6t + 1)(-1+t)^4 X^2.
\end{aligned}
$$

Setting this polynomial equal to zero gives us the desired equation.

3.2.2 Avoiding Peak and Valley Heights in Infinite Sets

Let A and B be two sets of arithmetic progressions $ar + b$ for non-negative integers a and b. Slight modifications to the procedure `fAB(A,B,t,X)` give us the procedure `fABr(A,B,r,t,X)`, which outputs the polynomial $F(t, X)$ such that $F(t, X) = 0$ is satisfied by the generating function for the sequence of Motzkin paths avoiding peak heights in A and valley heights in B.

For example,

```
fABr({2*r+1},{2*r+1},r,t,X)
```

outputs

$$(-1+t)^2 + (-1+t)^3 X + t^4 X^2.$$

Thus, the generating function of the sequence enumerating the Motzkin paths avoiding odd peak and valley heights satisfies the equation

$$(-1+t)^2 + (-1+t)^3 X + t^4 X^2 = 0.$$

3.2.3 Avoiding Upward-Run Lengths, Downward-Run Lengths, and Flat-Run Lengths in Finite Sets

Let C, D, and E be finite sets of positive non-negative integers. Here, we want to find the generating function $f_{C,D,E}$ of the sequence of Motzkin paths avoiding

- upward-runs with lengths in C,
- downward-runs with lengths in D, and
- flat-runs with lengths in E.

Let $h_{C,C_1,D,D_1,E,E_1,E_2}(t)$ weight enumerate Motzkin paths such that

- the initial run is not an upward-run with length in C_1 nor a flat-run with length in E_1,
- the initial run is an upward-run if $0 \in C_1$, and a flat-run if $0 \in E_1$,
- the final run is not a downward run with length in D_1 nor a flat-run with length in E_2,
- the final run is a downward-run if $0 \in D_1$, and a flat-run if $0 \in E_2$,
- all remaining upward-run lengths are not in C,
- all remaining downward-run lengths are not in D, and

- all remaining flat-run lengths are not in E.

Let $\mathcal{P}$ denote the set of such paths. For any path P in $\mathcal{P}$, P either leaves the $x-$axis no more than once or it can be uniquely written as

$$P = P_1P_2P_3,$$

where

- P_1 is a path that leaves the $x-$axis no more than once and has the same restrictions as paths in $\mathcal{P}$ except it ends in a downward-run avoiding lengths in D,
- P_2 is a path avoiding upward-runs with lengths in C, downward-runs with lengths in D, and flat-runs with lengths in E, and
- P_3 is a path that leaves the $x-$axis no more than once and has the same restrictions defined in $\mathcal{P}$ except it begins with an upward run avoiding lengths in C.

Let $H_{C,C_1,D,D_1,E,E_1,E_2}(t)$ enumerate the Motzkin paths counted by $h_{C,C_1,D,D_1,E,E_1,E_2}(t)$ that leave the $x-$axis no more than once.Then we have

$$\begin{aligned} h_{C,C_1,D,D_1,E,E_1,E_2}(t) =& H_{C,C_1,D,D_1,E,E_1,E_2}(t) \\ &+ H_{C,C_1,D,D\cup\{0\},E,E_1,E_2}(t)h_{C,C,D,D,E,E,E}(t)H_{C,C\cup\{0\},D,D_1,E,E_1,E_2}(t). \end{aligned}$$

Note that we will never have $0 \in C_1$ and $0 \in E_1$ or $0 \in D_1$ and $0 \in E_2$, since the former statement says that the path starts with both an up-step and a flat-step, and the latter states that the path ends with both a down-step and a flat-step. To get the desired system of equations and find the children of the set $\mathcal{P}$ of paths weight-counted by $H_{C,C_1,D,D_1,E,E_1,E_2}$, let P be any path in $\mathcal{P}$ and consider the following cases.

Case 1: If $0 \in E_1$, then P begins with a flat-step. Let $E_1' = \{e-1|e \in E_1\backslash\{0\}\}$. Then we can write

$$P = FP_1,$$

where P_1 is a path weight-counted by $H_{C,C,D,D_1,E,E_1',E_2}(t)$. Hence,

$$H_{C,C_1,D,D_1,E,E_1,E_2}(t) = t \cdot H_{C,C,D,D_1,E,E_1',E_2}(t)$$

Case 2: If $0 \notin E_1$ and $0 \in E_2$ then P ends with a flat-step. Let $E_2' = \{e-1|e \in E_2\backslash\{0\}\}$, and write

$$P = P_1F,$$

where P_1 is a path weight-counted by $H_{C,C,D,D_1,E,E_1,E_2'}(t)$. This gives us

$$H_{C,C_1,D,D_1,E,E_1,E_2}(t) = t \cdot H_{C,C,D,D_1,E,E_1,E_2'}(t)$$

Case 3: If $0 \notin E_1$, $0 \notin E_2$, $0 \in C_1$, and $0 \in D_1$, then P starts with an up-step and ends with a down-step. Letting $C_1' = \{c-1|c \in C_1\backslash\{0\}\}$ and $D_1' = \{d-1|d \in D_1\backslash\{0\}\}$, we can write

$$P = UP_1D,$$

where P_1 is a path weight-counted by $h_{C,C_1',D,D_1',E,E_1,E_2}(t)$. (Note that P_1 is able to return to the height it begins at more than once.) Thus,

$$H_{C,C_1,D,D_1,E,E_1,E_2}(t) = t^2 \cdot h_{C,C_1',D,D_1',E,E,E}(t)$$

Case 4: If $0 \notin E_1$, $0 \notin E_2$, $0 \notin C_1$, and $0 \notin D_1$ then P is either the empty path, starts with an up-step, or starts with a flat-step. We therefore get

$$H_{C,C_1,D,D_1,E,E_1,E_2}(t) = H_{C,C_1\cup\{0\},D,D_1,E,E,E_2,}(t) + H_{C,C_1,D,D_1,E,E_1\cup\{0\},E_2}(t) + 1$$

Case 5: If $0 \notin E_1$, $0 \notin E_2$, $0 \notin C_1$, and $0 \in D_1$, then P is non-empty and starts with either an up-step or a flat-step. Hence,

$$H_{C,C_1,D,D_1,E,E_1,E_2}(t) = H_{C,C_1\cup\{0\},D,D_1,E,E,E_2}(t) + H_{C,C_1,D,D_1,E,E_1\cup\{0\},E_2}(t)$$

Case 6: If $0 \notin E_1$, $0 \notin E_2$, $0 \in C_1$, and $0 \notin D_1$, then P is non-empty and ends with either a down-step or a flat-step. Thus,

$$H_{C,C_1,D,D_1,E,E_1,E_2}(t) = H_{C,C_1,D,D_1\cup\{0\},E,E,E_2}(t) + H_{C,C_1,D,D_1,E,E_1,E_2\cup\{0\}}(t).$$

We again generate finitely many descendants from the original set, as we will eventually remove all of the elements in C, D, and E. We also have as many equations as variables. Note that these polynomials generate an ideal. Since any basis will give the same set of solutions, we can look at the reduced Gröbner basis of the generated ideal.

Choosing the correct monomial ordering (namely, pure lexicographic order) will allow us to ensure that the smallest element of the reduced Gröbner basis is in the form to most easily find the desired equation satisfied by the generating function. This is due to the following theorem. (Here, we assign each descendant found in our system of equations a variable x_i, and let x_n be the variable representing the original family of restricted Motzkin paths. We do not consider x as one of these variables.)

Theorem 3.1 (Elimination Theorem)**.** *If G is a Gröbner basis for I with respect to* *lex order* $x_1 > x_2 > \cdots > x_n$, *then*

$$G_\ell = G \cap k[x_{\ell+1}, ..., x_n]$$

is a Gröbner basis of the ℓ*-th elimination ideal* $I_\ell = I \cap k[x_{\ell+1}, ..., x_n]$.

If $f(x)$ denotes the generating function of the sequence enumerating our original family of restricted Motzkin paths, then $x_n = f(t)$ is a partial solution to our system of equations represented by I. By the Elimination Theorem, if q denotes the smallest polynomial of the reduced Gröbner basis, then either $q \in G_{n-1}$ or $I_{n-1} = \langle 0 \rangle$. $I_{n-1} = \langle 0 \rangle$, however, contradicts the existence of the desired nonzero polynomial $F(t, X)$. Thus, $q \in G$ is a polynomial in terms of t and x_n and is zero when $x_n = f(t)$. Factoring q completely, we can write

$$q = q_1^{d_1} \ldots q_k^{d_k},$$

where $d_i \geq 1$. If $k = 1$, then we are done and $F(t, X) := q_1$. Otherwise, $x_n = f(t)$ also satisfies $q_i = 0$ for one of the factors q_i. We can then use the first m terms, where m is sufficiently large, of the sequence of interest to determine which factor is the desired q_i. We thereby get the desired polynomial $F(t, X) := q_i$.

This process is implemented in `fCDE(C,D,E,t,X)`.

```
fCDE({1,2,3},{},{},t,X)
```

outputs

$$1 + (-t^2 + t - 1)X - t^2(t - 1)X^2 + X^4 t^8 + X^5 t^9,$$

and

```
fCDE({},{1},{1},t,X)
```

outputs

$$t^2 - t + 1 + (-t^4 + t^3 - t^2 + t - 1)X + t^2(t^4 - t^3 + t^2 - t + 1)X^2 + X^3t^6.$$

Thus, when $X = \sum_{n=0}^{\infty} a(n)t^n$, where $a(n)$ is the number of Motzkin paths of length n avoiding upward runs of lengths 1, 2, and 3,

$$1 + (-t^2 + t - 1)X - t^2(t - 1)X^2 + X^4t^8 + X^5t^9 = 0.$$

If $a(n)$ is the number of Motzkin paths of length n avoiding downward-runs and flat-runs of length 1, then

$$t^2 - t + 1 + (-t^4 + t^3 - t^2 + t - 1)X + t^2(t^4 - t^3 + t^2 - t + 1)X^2 + X^3t^6 = 0.$$

3.2.4 Avoiding Upward-Run Lengths, Downward-Run Lengths, and Flat-Run Lengths in Infinite Sets

Suppose C, D, and E are sets of arithmetic progressions $ar + b$ for non-negative integers a and b. Through slight modifications to `fCDE(C,D,E,t,X)`, we get the procedure `fCDEr(C,D,E,r,t,X)`. `fCDEr(C,D,E,r,t,X)` finds the desired polynomial $F(t, X)$ that is zero when X is the generating function for the sequence of Motzkin paths avoiding upward-run lengths in C, downward-run lengths in D, and flat-run lengths in E. Running

```
fCDEr({2*r+1},{2*r+1},{2*r+1},r,t,X)
```

tells us that when X is the generating function of the sequence enumerating Motzkin paths avoiding upward, downward, and flat runs of odd length, we have

$$1 + (t-1)(t+1)X + X^2t^4 = 0$$

To get the equation satisfied by the generating function of the sequence enumerating Motzkin pats avoiding upward runs of odd lengths and flat-runs of positive even length, input

```
fCDEr({2*r+1},{},{2*r+2},r,t,X).
```

This tells us that our desired equation is

$$t^2 - t - 1 - (t-1)(t+1)X + t^4(t^2 - t - 1)X^3 = 0.$$

Further Study

Using similar approaches, we can create ways to automate counting of other objects. The approach of using numeric dynamic programming can efficiently generate many terms of the desired sequence. Guessing the algebraic equation, however, will not always work well. Thus, for larger problems, we need to use symbolic dynamic programming instead. Here, we identify recursive relations for the sets of relevant objects. Then, we use this system of equations to find an equality solved by the weight-enumerator of the set of combinatorial objects of interest.

Chapter 4

Automated Counting of Generalized Dyck Paths

In this chapter, we discuss how to enumerate generalized Dyck paths using symbolic computation. The Maple code discussed in this chapter is found in the package `GDW.txt`, written by AJ Bu and Doron Zeilberger. The package, along with sample outputs, can be found at

https://sites.math.rutgers.edu/ zeilberg/mamarim/mamarimhtml/area.html.

Definition 4.1. A *generalized Dyck path* is a path in the $xy-$plane from the origin $(0,0)$ to $(n,0)$ with an arbitrary set of atomic steps and that never goes below the $x-$axis.

Given a set of integers S, let $\mathcal{P}$ denote the set of generalized Dyck paths with steps in $\{(1,s) : s \in S\}$. To count the number of such paths, we have the following weight enumerator

$$f(t) := \sum_{P \in \mathcal{P}} t^{Length(P)}.$$

Equivalently, the weight enumerator is the ordinary generating function

$$\sum_{n=0}^{\infty} a_S(n) t^n$$

of the sequence $\{a_S(n)\}_{n=0}^{\infty}$, where $a_S(n)$ is the number of paths in the $xy-$plane from $(0,0)$ to $(n,0)$ with steps in $\{(1,s) : s \in S\}$.

Note that if S consists of only positive integers (or only negative integers), then no such path can exist. In this case, clearly $f(t) = 1$. Thus, for things to be non-trivial, S must have at least one positive member and at least one negative member.

4.1 Generating the Weight Enumerator

Given a set of legal steps, we can find the polynomial $F(t, X)$ such that $F(t, f(t)) = 0$, where $f(t)$ is the generating function for the sequence counting generalized Dyck paths of length n with steps in S. As was done in Chapter 3 Section 3, we generate a system of equations that will allow us to solve for the desired weight enumerator. First, we introduce the some notation for given non-negative integers a and b and a set of integers S.

$\mathcal{P}_{a,b} =$ the set of generalized Dyck paths with a set of steps given by S that start at $(0, a)$ and end at height b,

$f[a, b](t) =$the desired weight-enumerator for the paths in $\mathcal{P}_{a,b}$.

We also want to look at the subset of paths that only touch the $x-$axis at an endpoint, so we define the following.

$$\begin{aligned} \mathcal{Q}_{a,b} =& \text{the subset of } \mathcal{P}_{a,b} \text{ that contains all non-empty paths that stay strictly} \\ & \text{above the } x-\text{axis, except at an endpoint if } a=0 \text{ or } b=0, \\ g[a,b](t) =& \text{the desired weight-enumerator for the paths in } \mathcal{Q}_{a,b}. \end{aligned}$$

We start with $\mathcal{P}_{0,0}$, but we generate more "children" sets $\mathcal{P}_{a,b}$ and $\mathcal{Q}_{a,b}$ with various starting and ending heights. The children are the sets such that any element of the original set can be written in the form of concatenating certain steps with paths in the children sets. We then use the enumerating function for the children to get the enumerator for the original set. Sometimes, we will replace a child set with one that has the same number of elements but is easier to work with. For example, decreasing the height of the paths in $\mathcal{Q}_{1,1}$ by 1 gives a bijection with $\mathcal{P}_{0,0}$, so we can use $f[0,0](t)$ instead of $g[1,1](t)$. We then repeat this whole process with each child set until no more children are produced. Assigning different variables to each of these sets gives us our system of equations.

Forming the equations for $f[a,b](t)$

This section discusses how to find the equation for $f[a,b](t)$ for given non-negative integers a and b. This process is implemented by the procedure `MakeEqF(f,g,t,a,b,S)` in the Maple package `GDW.txt`. To form the equations for each $f[a,b](t)$, consider the following cases:

Case 1: Suppose $a>0$ and $b>0$.

Since the paths in $\mathcal{Q}_{a,b}$ never touch the $x-$axis, they must always have a height of at least 1. Lowering the paths in $\mathcal{Q}_{a,b}$ by 1 unit gives a bijection with the paths in $\mathcal{P}_{a-1,b-1}$. These paths are therefore enumerated by $f[a-1,b-1](t)$. The remaining paths in $\mathcal{P}_{a,b}$ must touch the $x-$axis at least once. For any such path P, we can uniquely rewrite P as

$$P = QP_1,$$

where $Q \in \mathcal{Q}_{a,0}$ and $P_1 \in \mathcal{P}_{0,b}$. Moreover, it is clear that for any $Q' \in \mathcal{Q}_{a,0}$ and $P' \in \mathcal{P}_{0,b}$, we have $Q'P' \in \mathcal{P}_{a,b}$. Thus, $\mathcal{P}_{a,b} \setminus \mathcal{Q}_{a,b}$ has the following grammar

$$\mathcal{P}_{a,b} \setminus \mathcal{Q}_{a,b} = \mathcal{Q}_{a,0}\mathcal{P}_{0,b}$$

and therefore is weight counted by $g[a,0](t) \cdot f[0,b](t)$. Thus,

$$f[a,b](t) \ = \ g[a,0](t) \cdot f[0,b](t) \ + \ f[a-1,b-1](t).$$

Case 2: If $a > 0$ and $b = 0$ then the paths in $\mathcal{P}_{a,0}$ must touch $x-$axis for a first time. It is therefore easy to see $\mathcal{P}_{a,0}$ satisfies the following grammar

$$\mathcal{P}_{a,0} = \mathcal{Q}_{a,0}\mathcal{P}_{0,0}.$$

Thus,

$$f[a,0](t) = g[a,0](t) \cdot f[0,0](t).$$

Case 3: Suppose $a = 0$ and $b > 0$. Now, we observe that the paths in $\mathcal{P}_{0,b}$ must hit the $x-$axis a final time. It is therefore obvious that $\mathcal{P}_{0,b}$ satisfies the following grammar

$$\mathcal{P}_{0,b} = \mathcal{P}_{0,0}\mathcal{Q}_{0,b},$$

and so

$$f[0,b](t) = f[0,0](t) \cdot g[0,b](t).$$

Case 4: Suppose $a = 0$, $b = 0$, and $0 \notin S$. $\mathcal{P}_{0,0}$ contains the empty path, and any nonempty path $P \in \mathcal{P}_{0,0}$ must touch the $x-$axis for a first time. We can therefore derive the following grammar

$$\mathcal{P}_{0,0} = \{EmptyPath\} \cup \mathcal{Q}_{0,0}\mathcal{P}_{0,0}.$$

It follows that

$$f[0,0](t) = 1 + g[0,0](t) \cdot f[0,0](t).$$

Case 5: Suppose $a = 0$, $b = 0$, and $0 \in S$. Again, $\mathcal{P}_{0,0}$ contains the empty path. Let $\mathcal{R}$ denote the set of paths in $\mathcal{P}_{0,0}$ that begin with the step $F = (1, 0)$. It is easy to derive the grammar

$$\mathcal{R} = F\mathcal{P}_{0,0}.$$

Let $\mathcal{S}$ denote the set of non-empty paths in $\mathcal{P}_{0,0}$ that do not start with a flat step. Then, the elements of $\mathcal{S}$ must return to the $x-$axis for a first time, giving the grammar

$$\mathcal{S} = \mathcal{Q}_{0,0}\mathcal{P}_{0,0}.$$

Thus, we have

$$\begin{aligned}\mathcal{P}_{0,0} &= \{EmptyPath\} \cup \mathcal{R} \cup \mathcal{S}\\ &= \{EmptyPath\} \cup F\mathcal{P}_{0,0} \cup \mathcal{Q}_{0,0}\mathcal{P}_{0,0}.\end{aligned}$$

Hence,

$$f[0,0](t) = 1 + tf[0,0](t) + g[0,0](t) \cdot f[0,0](t).$$

Forming the equations for $g[a,b](t)$

We also need to find the desired equations for necessary $g[a,b](t)$. This process is implemented by the procedure `MakeEqG(f,g,t,a,b,S)` in the Maple package `GDW.txt`.

Let P be the subset of S consisting of the (strictly) positive members of S, and let N be the subset of S consisting of the (strictly) negative members of S, so if $0 \in S$ then

$$S = P \cup N \cup \{0\},$$

while, if $0 \notin S$ then

$$S = P \cup N.$$

Note, as seen in the previous section, any $g[a,b](t)$ will have at least one of a or b being 0. We only need to consider the following cases for $g[a,b](t)$.

Case 1: Suppose $a = 0$ and $b > 0$. Note that the set of possible first steps of any path in $\mathcal{Q}_{0,b}$ is $\{s_k = (1,k) : k \in P\}$. So,

$$\mathcal{Q}_{0,b} = \bigcup_{k \in P} s_k \mathcal{Q}_{k,b}.$$

Since the sub-path in $\mathcal{Q}_{k,b}$ must always have height of at least 1, shifting it down by 1 unit gives a bijection between $\mathcal{Q}_{k,b}$ and $\mathcal{P}_{k-1,b-1}$. Since the first step s_k has weight t, we have

$$g[0,b](t) = t \sum_{k \in P} f[k-1, b-1](t).$$

Case 2: Suppose $a > 0$ and $b = 0$. The set of legal final steps of any path in $\mathcal{Q}_{a,0}$ is

$\{s_\ell = (1, \ell) : \ell \in N\}$. Thus,

$$\mathcal{Q}_{a,0} = \bigcup_{\ell \in N} \mathcal{Q}_{a,-\ell} s_\ell.$$

Since the sub-path in $\mathcal{Q}_{a,-\ell}$ must always have height of at least 1, shifting it down by 1 unit gives a bijection between $\mathcal{Q}_{a,-\ell}$ and $\mathcal{P}_{a-1,-\ell-1}$. Thus,

$$g[a, 0](t) = t \sum_{\ell \in N} f[a-1, -\ell-1](t).$$

Case 3: Suppose $a = 0$ and $b = 0$. The set legal starting steps for the paths in $\mathcal{Q}_{0,0}$ are $\{s_k : k \in P\}$ and the legal final steps are $\{s_\ell : \ell \in N\}$. Therefore,

$$\mathcal{Q}_{0,0} = \bigcup_{k \in P} \bigcup_{\ell \in N} s_k \mathcal{Q}_{k,-\ell} s_\ell,$$

and any path in $\mathcal{Q}_{k,-\ell}$ always has height of at least 1. Thus, shifting the paths in $\mathcal{Q}_{k,-\ell}$ down by 1 unit gives a bijection between $\mathcal{Q}_{k,-\ell}$ and $\mathcal{P}_{k-1,-\ell-1}$. Thus,

$$g[0, 0](t) = t^2 \sum_{k \in P} \sum_{\ell \in N} f[k-1, -\ell-1](t).$$

4.1.1 Solving the system of equations

Now, we have a set of of variables (from the $f[a, b](t)$ and $g[a, b](t)$ that we found), and an equation for each of these variables. To get the list of equations and variables, use `MakeSysT(f,g,t,S)` from the Maple package `GDW.txt`. It returns the set of equations, followed by the set of quantities that participate.

For example

```
MakeSysT(f, g, t, {1, 2, -1, -2})[1];
```

outputs

$$\{f_{00} = f_{00}g_{00} + 1 \quad , \quad f_{01} = f_{00}g_{01} \quad , \quad f_{10} = g_{10}f_{00} \quad , \quad f_{11} = f_{01}g_{10} + f_{00},$$

$$g_{00} = t^2 f_{00} + t^2 f_{01} + t^2 f_{10} + t^2 f_{11} \quad , \quad g_{01} = t f_{00} + t f_{10} \quad , \quad g_{10} = t f_{00} + t f_{01}\} \quad ,$$

while

$$\mathtt{MakeSysT}(\mathtt{f}, \mathtt{g}, \mathtt{t}, \{1, 2, -1, -2\})[2];$$

outputs the set of quantities

$$\{f_{00} \quad , \quad f_{01} \quad , \quad f_{10} \quad , \quad f_{11} \quad , \quad g_{00} \quad , \quad g_{01} \quad , \quad g_{10}\} \quad .$$

We now assign a variable x_i to each descendent, and let x_n denote the set of generalized Dyck paths. We look at the ideal generated by each of the polynomials from the equations. Using the Elimination Theorem and Gröbner bases, we can ensure that the smallest element in the reduced Gröbner basis is in the form to most easily find $F(t, X) = 0$. A more detailed explanation of this application of the Elimination Theorem was given in Chapter 3 Section 2.3. This final step is implemented in the procedure `EqGFt(S,X,t)`.

For example,

```
EqGFt(1,2,-1,-2,X,t)
```

outputs

$$1 + (-2t - 1)X + t(3t + 2)X^2 - t^2(2t + 1)X^3 + X^4 t^4.$$

We can also use similar methods to count strict generalized Dyck paths, i.e. paths that never touch the x-axis except at the endpoints. This is implemented by the procedure `EqGFtS(S,X,t)`.

For example,

```
EqGFtS(1,2,-1,-2,X,t)
```

outputs

$$(-t-1+X)(X^4+2X^3t+3X^2t^2+2Xt^3+t^4-3X^3-4X^2t-5Xt^2-2t^3+3X^2+2Xt+2t^2-X).$$

Chapter 5

The Sum of the Areas under Dyck and Motzkin paths and Their Powers

In this chapter, we are interested in the generating functions for the sum of the areas under generalized Dyck paths, with a focus on Dyck and Motzkin paths. For example, the following are some Motzkin paths of length 4.

$$UDUD \qquad UFFD \qquad UFDF \qquad FUDF \qquad FFFF.$$

The areas of these paths are 2, 4, 3, 1, and 0, respectively.

The bivariate weight enumerator for Motzkin paths with length n and area m satisfies the following functional equation

$$M(t,q) = 1 + tM(t,q) + t^2qM(t,q)M(t,q).$$

To prove this, let $\mathcal{M}$ denote the set of all Motzkin paths. Note that any path in $\mathcal{M}$ must fall into exactly one of the following cases – the empty path, Motzkin paths that start with a flat step, or Motzkin paths that start with an up step.

If $M \in \mathcal{M}$ is the empty path, then it clearly has both area and length 0. Thus the bivariate weight enumerator is

$$m_0(t, q) = 1$$

If M begins with a flat step F, then we can write

$$M = FM_0,$$

where M_0 must also be a Motzkin path with the same area as M, since it still starts at height 0. Thus, the bivariate weight enumerator for this case of Motzkin paths is

$$m_1(t, q) = tM(t, q)$$

If M begins with the step U, then let D denote the first time M returns to the $x-$axis and write

$$M = UM_1DM_0.$$

M_1 must be a Motzkin path shifted to height 1, and M_0 is a Motzkin path starting at height 0. Since M_0 begins at height 0, the area under the Motzkin path M_0 is the same as the area under the portion of M it represents. Since M_1 is shifted to height 1, however, every step in M_1 has one more unit block below it. Thus, every step t in M_1 must be replaced with qt to get the correct area for that portion of M. Since the extra U and D steps give a combined area of 1, the bivariate weight enumerator for

Motzkin paths beginning with an up step is

$$m_2(t,q) = qt^2 M(qt,q)M(t,q),$$

resulting in the desired weight enumerator for all Motzkin paths.

Similarly, the bivariate weight enumerator for Dyck paths with length n and area m satisfies the following functional equation

$$D(t,q) = 1 + t^2 q D(qt,q)D(t,q).$$

Accompanying Maple Packages

This chapter presents procedures in the Maple packages `qEW.txt` and `qGDW.txt`. The Maple package `qEW.txt` and some sample outputs can be found at

`https://ajbu1.github.io/Papers/MotzArea/MotzArea.html` .

The Maple package `GDW.txt`, along with sample outputs, can be found at

`https://sites.math.rutgers.edu/ zeilberg/mamarim/mamarimhtml/area.html`

5.1 Enumerating Generalized Dyck Paths of Length $n = 0, \ldots, K$ with Steps in S by Area

The procedure `qnwdK(S,K,q)` in the Maple package `qEW.txt` uses dynamic programming to find the enumerating function for the area of generalized Dyck paths with steps in `S` of length $n = 0, \ldots, K$. This procedure can be used to check the findings presented later in this chapter.

First, consider $\mathcal{W}_{m,n}$, the set of paths of length $n \geq 0$ with steps in S that end at height $m \geq 0$ and never have negative height. Let $A_{m,n}(q)$ be the enumerating function for the area of paths in $\mathcal{W}_{m,n}$.

Clearly, for $n = 0$, the empty path gives an area of 0. Thus, the enumerating function is

$$A_{m,0}(q) = 1.$$

For $n = 1$, the only path that can end at height m is the single step $\{(1, m)\}$, which has area $\frac{m}{2}$. Thus,

$$A_{m,1}(q) = \begin{cases} q^{\frac{m}{2}}, & (1, m) \in S \\ 0, & (1, m) \notin S. \end{cases}$$

For $n > 1$, consider each possible final step for any path in $\mathcal{W}_{m,n}$. A step $s \in S$ can be the last step if $m - s \geq 0$ and there exists a path W of length $n - 1$ with steps in S that ends at height $m - s$ and never has a negative height. In other words,

$$\mathcal{W}_{m,n} = \{Ws | s \in S,\ m - s \geq 0,\ W \in \mathcal{W}_{m-s,n-1}\}.$$

The area under the last step $(1, s)$ is $\frac{2m-s}{2}$. Thus, the weight enumerator for the area of paths in $\mathcal{W}_{m,n}$ is

$$A_{m,n}(q) = \sum_{\substack{s \in S \\ m-s \geq 0}} q^{\frac{2m-s}{2}} A_{m-s,n-1}(q).$$

This process is implemented in the procedure `qnmwd(S,n,m,q)`, which is then used in `qnwdK(S,K,q)`. For example, looking at Motzkin paths,

```
qnwdK({[1,1],[1,0],[1,-1]},5,q)
```

outputs

$$[1, 1, q+1, q^2+2q+1, q^4+q^3+3q^2+3q+1, q^6+2q^5+3q^4+4q^3+6q^2+4q+1]\ .$$

Note that, to avoid negative height, any Motzkin path must end with $D = (1, -1)$ or $F = (1, 0)$. Since the paths end at height 0, the area under these steps are $\frac{1}{2}$ and 0, respectively. Thus,

$$A_{0,n} = q^{\frac{1}{2}} A_{1,n-1} + A_{0,n-1}.$$

Breaking down the algorithm described above to find the first four terms of this outputted list,

- The only path of length 1 is $[F] = [(1, 0)]$, so

$$A_{0,1}(q) = 1.$$

- Since $\mathcal{W}_{1,1} = \{[U]\} = \{[(1, 0)]\}$, it follows that $A_{1,1}(q) = q^{\frac{1}{2}}$. Thus,

$$\begin{aligned} A_{0,2} &= q^{\frac{1}{2}} A_{1,1}(q) + A_{0,1}(q) \\ &= q + 1. \end{aligned}$$

- For paths of length 3 ending with $D = (1, -1)$, note that

$$\mathcal{W}_{1,2} = \{FU, UF\} = \{[(1, 0), (1, 1)], [(1, 1), (1, 0)]\},$$

and so $A_{1,2} = q^{\frac{1}{2}} + q^{\frac{3}{2}}$. Thus,

$$\begin{aligned} A_{0,3} &= q^{\frac{1}{2}} A_{1,2}(q) + A_{0,2}(q) \\ &= q^{\frac{1}{2}}(q^{\frac{1}{2}} + q^{\frac{3}{2}}) + q + 1 \\ &= q^2 + 2q + 1. \end{aligned}$$

- For paths of length 4, note that

$$\begin{aligned} \mathcal{W}_{1,3} &= \{FFU, FUF, UFF, UDU\} \\ &= \{[(1,0),(1,0),(1,1)],[(1,1),(1,-1),(1,1)],[(1,0),(1,1),(1,0)], \\ &\qquad [(1,1),(1,0),(1,0)]\}. \end{aligned}$$

Therefore, $A_{1,3} = q^{\frac{1}{2}} + 2q^{\frac{3}{2}} + q^{\frac{5}{2}}$, and

$$\begin{aligned} A_{0,4} &= q^{\frac{1}{2}} A_{1,3}(q) + A_{0,3}(q) \\ &= q^4 + 3q^2 + 3q + 1. \end{aligned}$$

5.2 Weighted Enumeration of Generalized Dyck Paths According to the Area

Now, we present a method analogous to the one presented in Chapter 4 but, in addition to enumerating generalized Dyck paths of length n, we also keep track of the area under the paths. Given a set of legal steps, we want to find the polynomial $F(t, X)$ such that $F(t, f(t, q)) = 0$, where $f(t, q)$ is the desired bi-variate weight enumerator.

Note that we have have the additional variable q, which we can treat as a parameter. Therefore, in this section, we will write $f[0,0](t)$ instead of $f[0,0](t,q)$. For given non-negative integers a and b and a set of integers S, let

$$\begin{aligned}
\mathcal{P}_{a,b} =& \text{ the set of generalized Dyck paths with a set of steps given by } S \text{ that} \\
& \text{start at } (0,a) \text{ and end at height } b, \\
f[a,b](t) =& \text{the desired weight-enumerator for the paths in } \mathcal{P}_{a,b}, \text{ where the weight} \\
& \text{of a path } P \text{ is } t^{Length(P)} \cdot q^{AreaUnder(P)}, \\
\mathcal{Q}_{a,b} =& \text{the subset of } \mathcal{P}_{a,b} \text{ that contains all non-empty paths that stay strictly} \\
& \text{above the } x-\text{axis, except at an endpoint if } a=0 \text{ or } b=0, \\
g[a,b](t) =& \text{the desired weight-enumerator for the paths in } \mathcal{Q}_{a,b}, \text{ where the weight} \\
& \text{of a path } P \text{ is } t^{Length(P)} \cdot q^{AreaUnder(P)}.
\end{aligned}$$

Again, we start with $\mathcal{P}_{0,0}$ and rewrite it by concatenating specific steps with paths of "children" sets, which will have the form $\mathcal{P}_{a,b}$ and $\mathcal{Q}_{a,b}$ with various starting and ending heights. We then use the enumerating function for the children to get the enumerator for the original set. We repeat this whole process with each descendent set until no more children are produced.

Note that our children sets are the same as in Chapter 4, but we need to adjust the bi-variate weight-enumerator to account for any changes in area. Additionally, the generating function for the sum of the areas of all legal walks of length n will be given by

$$\left[\frac{d}{dq} f[0,0](t)\right]_{q=1}.$$

Forming the Functional Equations for $f[a,b](t)$

The procedure `qMakeEqF(f,g,t,q,a,b,S)` in the Maple package `GDW.txt` generates the functional equation for $f[a,b](t)$, where a and b are given non-negative integers. To form the functional equation, we need to look at the following cases:

Case 1: Suppose $a > 0$ and $b > 0$.

Since the paths in $\mathcal{Q}_{a,b}$ never touch the x-axis, they must always have a height of at least 1. Thus, lowering these paths by 1 unit gives a bijection between $\mathcal{Q}_{a,b}$ and $\mathcal{P}_{a-1,b-1}$. While the number of such paths are equal, this mapping changes the area under the path. Note that for each step in the path, the area is reduced by 1 unit^2. Thus, the paths in $\mathcal{Q}_{a,b}$ are bi-weight-enumerated by $f[a-1,b-1](qt)$.

The remaining paths in $\mathcal{P}_{a,b}$ must touch the $x-$axis at least once (and therefore for a first time). As shown in Chapter 4, the following grammar is clear

$$\mathcal{P}_{a,b} \setminus \mathcal{Q}_{a,b} = \mathcal{Q}_{a,0}\mathcal{P}_{0,b}.$$

Thus, we have the following equation

$$f[a,b](t) = g[a,0](t) \cdot f[0,b](t) + f[a-1,b-1](qt).$$

Case 2: Suppose $a > 0$ and $b = 0$. Each path in $\mathcal{P}_{a,0}$ hits the $x-$axis for a first time. Thus, it is easy to prove that

$$\mathcal{P}_{a,b} = \mathcal{Q}_{a,0}\mathcal{P}_{0,0},$$

and

$$f[a,0](t) \ = \ g[a,0](t) \cdot f[0,0](t).$$

Case 3: Similarly, for $a = 0$ and $b > 0$, the paths in $\mathcal{P}_{0,b}$ must hit the $x-$axis for a final time. We therefore have the grammar

$$\mathcal{P}_{0,b} = \mathcal{P}_{0,0}\mathcal{Q}_{0,b},$$

and so

$$f[0,b](t) = f[0,0](t) \cdot g[0,b](t).$$

Case 4: If $a = 0$ and $b = 0$ and $0 \notin S$, then the walks in $\mathcal{P}_{0,0}$ must either be the empty path, which has weight 1 since its area and length are both 0, or it must meet the $x-$axis for the first time. Hence,

$$\mathcal{P}_{0,0} = \{EmptyPath\} \cup \mathcal{Q}_{0,0}\mathcal{P}_{0,0},$$

and

$$f[0,0](t) = 1 + g[0,0](t) \cdot f[0,0](t).$$

Case 5: Suppose $a = 0$, $b = 0$, and $0 \in S$. Again, $\mathcal{P}_{0,0}$ contains the empty path. Let $\mathcal{R}$ denote the set of paths in $\mathcal{P}_{0,0}$ that begin with the step $F = (1,0)$. It is easy to derive the grammar

$$\mathcal{R} = F\mathcal{P}_{0,0}.$$

Let $\mathcal{S}$ denote the set of non-empty paths in $\mathcal{P}_{0,0}$ that do not start with a flat step. Then, the elements of $\mathcal{S}$ must return to the $x-$axis for a first time, giving the grammar

$$\mathcal{S} = \mathcal{Q}_{0,0}\mathcal{P}_{0,0}.$$

Thus, we have

$$\begin{aligned}\mathcal{P}_{0,0} &= \{EmptyPath\} \cup \mathcal{R} \cup \mathcal{S} \\ &= \{EmptyPath\} \cup F\mathcal{P}_{0,0} \cup \mathcal{Q}_{0,0}\mathcal{P}_{0,0}.\end{aligned}$$

Hence,

$$f[0,0](t) = 1 + tf[0,0](t) + g[0,0](t) \cdot f[0,0](t).$$

Forming the Functional Equations for $g[a,b](t)$

We also need to set up equations for $g[a,b](t)$ for those (a,b) that would be required. As seen in the previous section, at least one of a or b will be 0. Let P be the subset of S consisting of the (strictly) positive members of S, and let N be the subset of S consisting of the (strictly) negative members of S. Now, consider the following cases for $g[a,b](t)$.

Case 1: Suppose $a = 0$ and $b > 0$. Note that the set of possible first steps of any path in $\mathcal{Q}_{0,b}$ is $\{s_k = (1,k) : k \in P\}$. So,

$$\mathcal{Q}_{0,b} = \bigcup_{k \in P} s_k \mathcal{Q}_{k,b}.$$

The step s_k has length 1 and area $\frac{k}{2}$, so its weight is $tq^{\frac{k}{2}}$. Moreover, since the sub-path in $\mathcal{Q}_{k,b}$ must always have height of at least 1, shifting it down by 1 unit gives a bijection between $\mathcal{Q}_{k,b}$ and $\mathcal{P}_{k-1,b-1}$. This mapping however reduces the area by 1 unit^2 for each step in the path. Thus,

$$g[0,b](t) = t \sum_{k \in P} q^{k/2} f[k-1, b-1](qt).$$

Case 2: Suppose $a > 0$ and $b = 0$. The set of legal final steps of any path in $\mathcal{Q}_{a,0}$ is $\{s_\ell = (1, \ell) : \ell \in N\}$. Thus,

$$\mathcal{Q}_{a,0} = \bigcup_{\ell \in N} \mathcal{Q}_{a,-\ell} s_\ell.$$

The step s_ℓ has weight $tq^{\frac{\ell}{2}}$, and shifting the paths in $\mathcal{Q}_{a,-\ell}$ down by 1 unit gives a bijection between $\mathcal{Q}_{a,-\ell}$ and $\mathcal{P}_{a-1,-\ell-1}$. Adjusting for the change in area, we have

$$g[a,0](t) = t \sum_{\ell \in N} q^{-\ell/2} f[a-1, b-j-1](qt).$$

Case 3: Suppose $a = 0$ and $b = 0$. The set legal starting steps for the paths in $\mathcal{Q}_{0,0}$ are $\{s_k : k \in P\}$ and the legal final steps are $\{s_\ell : \ell \in N\}$. Therefore,

$$\mathcal{Q}_{0,0} = \bigcup_{k \in P} \bigcup_{\ell \in N} s_k \mathcal{Q}_{k,-\ell} s_\ell,$$

and any path in $\mathcal{Q}_{k,-\ell}$ always has height of at least 1. Thus, shifting the paths in $\mathcal{Q}_{k,-\ell}$ down by 1 unit gives a bijection between $\mathcal{Q}_{k,-\ell}$ and $\mathcal{P}_{k-1,-\ell-1}$. Thus,

$$g[0,0](t) = t^2 \sum_{k \in P} \sum_{\ell \in N} q^{k/2-\ell/2} f[k-1, -\ell-1](qt).$$

The above process is implemented in procedure `qMakeEqGt(f,g,t,q,a,b,S)` in the Maple package `GDW.txt`.

Solving the System of Functional Equations

The procedure `qMakeSysT(f,g,t,q,S)` in the Maple package `GDW.txt` outputs the full system of functional equations, followed by the quantities that feature in them. For example, to get the system of equations for the generalized Dyck paths with steps in $S = \{2,1,0,-1,-2\}$, typing

```
qMakeSyst(f,g,t,q, { 2,1,0,-1,-2 } )[1];
```

outputs

$$\{f_{01}(t) - f_{00}(t)\, g_{01}(t) \quad , \quad f_{10}(t) - g_{10}(t)\, f_{00}(t) \quad ,$$

$$f_{11}(t) - f_{01}(t)\, g_{10}(t) - f_{00}(qt) \quad , \quad g_{01}(t) - t\sqrt{q}\, f_{00}(qt) - tqf_{10}(qt) \quad ,$$

$$g_{10}(t) - tqf_{01}(qt) - t\sqrt{q}\, f_{00}(qt)\,, \quad f_{00}(t) - f_{00}(t)\, g_{00}(t) - f_{00}(t)\, t - 1 \quad ,$$

$$g_{00}(t) - t^2 q^{\frac{3}{2}} f_{01}(qt) - t^2 q f_{00}(qt) - t^2 q^2 f_{11}(qt) - t^2 q^{\frac{3}{2}} f_{10}(qt)\}.$$

To see the set of featured quantities, type

$$\texttt{qMakeSyst}(\texttt{f}, \texttt{g}, \texttt{t}, \texttt{q}, \{2, 1, 0, -1, -2\})[2];$$

which outputs

$$\{f_{00}(t) \quad , \quad f_{00}(qt) \quad , \quad f_{01}(t) \quad , \quad f_{01}(qt) \quad , \quad f_{10}(t) \quad , \quad f_{10}(qt) \quad ,$$

$$f_{11}(t) \quad , \quad f_{11}(qt) \quad , \quad g_{00}(t) \quad , \quad g_{01}(t) \quad , \quad g_{10}(t)\} \quad .$$

After the computer finds the system of *functional* equations described above, we instruct it to find a system *algebraic* equations for the 'components' of the $f[a,b](t)$ (and we also need $g[a,b](t)$). To do this, we will use the the Taylor Series expansions about $q = 1$:

$$f[a,b](t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{d^n}{dq^n} f[a,b](t) \right]_{q=1} (q-1)^n.$$

We will also use the following lemma from Calculus:

Lemma 5.1. *If $f(t)$ is the formal power series of a single variable t, and q is another* *variable, then*

$$f(qt) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \left[\frac{d^n}{dt^n} f(t) \right] (q-1)^n.$$

Recall that the generating function for the sum of the areas of all legal walks of length n is $f_q[0,0](t)\Big|_{q=1}$. Therefore, we are really only interested in the powers up to $n = 1$. We will rewrite all of our $f[a,b](t)$ and $g[a,b](t)$ as

$$f[a,b](t) = \Big[f[a,b](t)\Big]_{q=1} + \Big[f_q[a,b](t)\Big]_{q=1}(q-1) + O((q-1)^2), \qquad \text{and}$$
$$g[a,b](t) = \Big[g[a,b](t)\Big]_{q=1} + \Big[g_q[a,b](t)\Big]_{q=1}(q-1) + O((q-1)^2).$$

We expand in powers of $q-1$, then collect terms, use Lemma 5.1, and get more equations by differentiating with respect to t each of these equations using implicit differentiation.

Note that this method works for higher order derivatives. Because of the extreme complexity, we decided to only implement this scheme for $k = 1$ to find the algebraic equation satisfied by the generating function for the sum of the areas.

This is implemented in procedure `qEqGFt(S,X,t)` . For example, to get the algebraic equation for the generating function for 'sum of areas' of the classical Dyck paths, type:

$$\texttt{qEqGFt}(\{1,-1\},\texttt{X},\texttt{t})$$

getting

$$t^2 \,-\, (4t^2-1)(2t^2-1)X \,+\, t^2(4t^2-1)^2\,X^2 \,=\, 0.$$

(This is A8549 of [22], `https://oeis.org/A008549`).

For Motzkin walks, typing

$$\texttt{qEqGFt}(\{1, 0, -1\}, \texttt{X}, \texttt{t})$$

gives

$$t^2 - (3t-1)(t+1)\left(t^2+2t-1\right)X + t^2(3t-1)^2(t+1)^2X^2 = 0.$$

(This is A57585 of [22], `https://oeis.org/A057585`).

For a more complicated example, to get the pure algebraic equation satisfied by the generating function for the 'sum of the areas under generalized Dyck paths with set of steps $\{[1,2],[1,1],[1,0],[1,-1],[1,-2]\}$, type:

$$\texttt{qEqGFt}(\{2, 1, 0, -1, -2\}, \texttt{X}, \texttt{t}),$$

getting, after less than a minute,

$$\begin{aligned}
&t^2\left(775t^4 - 1460t^3 + 1006t^2 - 264t + 24\right) \\
&\quad + (t-1)(5t-1)\left(425t^6 - 1520t^5 + 1527t^4 - 68t^3 - 282t^2 + 88t - 8\right)X \\
&\quad - t\left(150t^5 + 540t^4 - 889t^3 - 240t^2 + 228t - 32\right)(t-1)^2(5t-1)^2X^2 \\
&\quad - 2t^2(5t+4)\left(5t^3 - t^2 - 17t + 4\right)(t-1)^3(5t-1)^3X^3 + t^4(5t+4)^2(t-1)^4(5t-1)^4X^4 \\
&=0.
\end{aligned}$$

This is not (yet, May 15, 2023) in the OEIS. The first 30 terms are

$0, 0, 3, 18, 113, 636, 3487, 18656, 98429, 514012, 2664690, 13737758, 70522801,$

$360806214, 1840913908, 9371761174, 47621259557, 241601881822, 1224111502194,$

$6195045902854, 31321134873744, 158217553824544, 798622703316154, 4028438371631942,$

$20308239308212037, 102323623873153810, 515313296262175206, 2594054240062008690,$

$13053194513626873348, 65659889953142043370$

Note that the straight enumeration version is A104184 of [22], `https://oeis.org/A104184`.

Strict Generalized Dyck paths

To look at the area under strict generalized Dyck paths, i.e. paths that never touch the x-axis except at the endpoints, use procedure `qEqGFtS(S,X,t)`. For the algebraic equation for the generating function for the sum of the areas under **strict** classical Dyck paths, type

$$\texttt{qEqGFtS}(\{1,-1\},\texttt{X},\texttt{t});,$$

which gives us the equation

$$\left(4t^2-1\right)X+t^2 \;=\; 0.$$

Therefore,

$$X(t)=\frac{t^2}{1-4t^2} \quad ,$$

confirming, *purely automatically*, the following elegant proposition first discovered, and proved, in [27].

Proposition 5.1. *(Shapiro, Rogers, and Woan) The sum of the areas of the strict*

Dyck paths of length $2n$ *is* 4^{n-1}.

Looking at strict Motzkin paths, typing

$$\texttt{qEqGFtS}(\{\texttt{1}, \texttt{0}, -\texttt{1}\}, \texttt{X}, \texttt{t});$$

outputs

$$\left(3t^2 + 2t - 1\right) X + t^2 \, = \, 0.$$

Thus,

$$X(t) = \frac{t^2}{1 - 2t - 3t^2}.$$

This is A015518[n-1] of [22] (see `https://oeis.org/A015518`). This sequence has numerous combinatorial interpretations, but so far, the connection to the sum of the areas under strict Motzkin paths escaped notice.

5.3 Perturbation Expansions of Solutions to Quadratic Functional Equations

Suppose that a function $f(t, q)$ satisfies the functional equation

$$f(t, q) = P(t, q) + Q(t, q) f(t, q) + R(t, q) f(t, q) f(qt, q),$$

for given bivariate polynomials $P(t, q)$, $Q(t, q)$, and $R(t, q)$. To find $f(t, q)$ up to degree k in t, first set $f_0(t, q) := 1$. For $i > 0$, let

$$f_i(t, q) = P(t, q) + Q(t, q) f_{i-1}(t, q) + R(t, q) f_{i-1}(qt, q) f_{i-1}(t, q),$$

and find $n > 0$ such that $f_n(t,q)$ and $f_{n+1}(t,q)$ agree up to degree k in t. Note that for any $i > n$, $f_n(t,q)$ and $f_i(t,q)$ will also agree up to degree k in t, and thus $f(t,q)$ and $f_n(t,q)$ will as well.

This process is implemented in the Maple package `qEW.txt` by the procedure `SeqF1(P,Q,R,q,t,K)`, which inputs bivariate polynomials P, Q, and R, variables q and t, and a non-negative integer K, and outputs $f(t,q)$ up to degree K in t. For example, the weight enumerator for the area under Dyck paths of lengths $k = 0,\ldots,8$ is found by

`SeqF1(1,0,t^2*q,q,t,8)`,

which outputs

$$1+t^2q+(q^4+q^2)t^4+(q^9+q^7+2q^5+q^3)t^6+(q^{16}+q^{14}+2q^{12}+3q^{10}+3q^8+3q^6+q^4)t^8.$$

The weight enumerator for the area under Motzkin paths of lengths $k = 0,\ldots,5$ is found by

`SeqF1(1,t,t^2*q,q,t,5)`,

which outputs

$$1+t+(q+1)t^2+(q^2+2q+1)t^3+(q^4+q^3+3q^2+3q+1)t^4+(q^6+2q^5+3q^4+4q^3+6q^2+4q+1)t^5.$$

Note that the coefficient of t^k and the k-th term of the list output by

`qnwdK([1,1],[1,0],[1,-1]),5,q)`

are equal, as desired. Using this method, the expression is found through the following calculations

$$
\begin{aligned}
f_0(t,q) &= 1\\
f_1(t,q) &= 1 + tf_0(t,q) + t^2qf_0(t,q)f_0(qt,q)\\
&= 1 + t + t^2q\\
f_2(t,q) &= 1 + tf_1(t,q) + t^2qf_1(t,q)f_1(qt,q)\\
&= 1 + t + (q+1)t^2 + (q^2+2q)t^3 + (q^4+2q^2)t^4 + (q^4+q^3)t^5 + \dots\\
f_3(t,q) &= 1 + tf_2(t,q) + t^2qf_2(t,q)f_2(qt,q)\\
&= 1 + t + (q+1)t^2 + (q^2+2q+1)t^3 + (q^4+q^3+3q^2+3q)t^4\\
&\quad + (q^6+2q^5+2q^4+3q^3+5q^2)t^5 + \dots\\
f_4(t,q) &= 1 + tf_3(t,q) + t^2qf_3(t,q)f_3(qt,q)\\
&= 1 + t + (q+1)t^2 + (q^2+2q+1)t^3 + (q^4+q^3+3q^2+3q+1)t^4\\
&\quad + (q^6+2q^5+3q^4+4q^3+6q^2+4q)t^5 + \dots\\
f_5(t,q) &= 1 + tf_4(t,q) + t^2qf_4(t,q)f_4(qt,q)\\
&= 1 + t + (q+1)t^2 + (q^2+2q+1)t^3 + (q^4+q^3+3q^2+3q+1)t^4\\
&\quad + (q^6+2q^5+3q^4+4q^3+6q^2+4q+1)t^5 + \dots\\
f_6(t,q) &= 1 + tf_4(t,q) + t^2qf_4(t,q)f_4(qt,q)\\
&= 1 + t + (q+1)t^2 + (q^2+2q+1)t^3 + (q^4+q^3+3q^2+3q+1)t^4\\
&\quad + (q^6+2q^5+3q^4+4q^3+6q^2+4q+1)t^5 + \dots
\end{aligned}
$$

Since $f_5(t,q)$ and $f_6(t,q)$ agree up to degree 5 in t, the procedure outputs

$$1+t+(q+1)t^2+(q^2+2q+1)t^3+(q^4+q^3+3q^2+3q+1)t^4+(q^6+2q^5+3q^4+4q^3+6q^2+4q+1)t^5.$$

5.4 Finding $\frac{d^k}{dq^k}\left[f(t,q)\right]\Big|_{q=1}$

If the weight enumerator of a set of paths is satisfied by the following functional equation

$$f(t,q) = P(t,q) + Q(t,q)f(t,q) + R(t,q)f(t,q)f(qt,q)$$

for some given bivariate polynomials $P(t,q)$, $Q(t,q)$, and $R(t,q)$, then plugging in $q=1$ gives

$$f(t,1) = P(t,1) + Q(t,1)f(t,1) + R(t,1)f(t,1)^2,$$

which we can use to solve for $f(t,1)$. The order n Taylor polynomial of $f(t,q)$ about $q=1$ satisfies

$$\begin{aligned}\sum_{k=0}^{n}\frac{(q-1)^k}{k!}f^{(k)}(t,1) = P + Q&\sum_{k=0}^{n}\frac{(q-1)^k}{k!}f^{(k)}(t,1)\\ + R&\sum_{k=0}^{n}\frac{(q-1)^k}{k!}f^{(k)}(t,1)\sum_{k=0}^{n}\frac{(q-1)^k}{k!}f^{(k)}(qt,1),\end{aligned}$$

where $f^{(k)}(t,q) := \frac{d^k}{dq^k}f(t,q)$. Looking at the coefficient of $(q-1)^k$, we can express $f^{(k)}(t,1)$ as the sum of derivatives $f^{(\ell)}(t,1)$ where $\ell < k$ and derivatives of functions of t with respect to t. Since we have an expression for $f(t,1)$, we can simply compute any order derivative with respect to t as well as $f_q(t,1)$. Thus, to find $f^{(n)}(t,1)$, we can repeat this process with the coefficient of $f^{(k)}(t,1)$ for $k = 1,\ldots,n$.

This process is implemented by the procedure `DerK(P,Q,R,q,t,K,f)`, which outputs a list whose k-th entry is $\frac{d^{k-1}}{dq^{k-1}}\left[f(t,q)\right]\Big|_{q=1}$. Rather than outputting algebraic equations as seen in Section 5.2, this procedure produces closed-form expressions in terms of radicals.

Motzkin Paths

As previously noted, the Motzkin paths satisfy the following functional equation

$$M(t,q) = 1 + tM(t,q) + t^2 qM(qt,q)M(t,q).$$

Solving this functional equation for $q = 1$, we get that

$$M(t,1) = \frac{1 - t + \sqrt{-3t^2 - 2t + 1}}{2t^2} \qquad \text{or} \qquad M(t,1) = \frac{1 - t - \sqrt{-3t^2 - 2t + 1}}{2t^2}.$$

Since only the second equation has a Taylor series expansion about $t = 0$, we know that this is $M(t,1)$. Now, for finding the first derivative, note that

$$\sum_{k=0}^{n} \frac{(q-1)^k}{k!} M^{(k)}(t,1) = 1 + t\sum_{k=0}^{n} \frac{(q-1)^k}{k!} M^{(k)}(t,1) \\ + qt^2 \sum_{k=0}^{n} \frac{(q-1)^k}{k!} M^{(k)}(t,1) \sum_{k=0}^{n} \frac{(q-1)^k}{k!} M^{(k)}(qt,1).$$

The coefficient of $q - 1$ on both sides give us

$$M_q(t,1) = tM_q(t,1) + t^2 M(t,1)\Big(tM_t(t,1) + 2M_q(t,1) + M(t,1)\Big).$$

Therefore,

$$M_q(t,1) = \frac{t^3 M(t,1)M_t(t,1) + t^2 M^2(t,1)}{1 - t - 2t^2 M(t,1)}.$$

Plugging in $M(t,1) = \frac{1-t-\sqrt{-3t^2-2t+1}}{2t^2}$, we get

$$M_q(t,1) = \frac{\left(t - 1 + \sqrt{-3t^2 - 2t + 1}\right)^2}{4t^2(-3t^2 - 2t - 1)}$$

To find $M^{(n)}(t,1)$, we can repeat this process with the coefficient of $M^{(k)}(t,1)$ for $k \leq n$.

In a little over 2 seconds,

```
DerK(1,t,t^2*q,q,t,10,f),
```

can output the list whose entries are $M^{(k)}(t,1) := \frac{d^k}{dq^k}\left[M(t,q)\right]|_{q=1}$ for $k = 0, \ldots, 10$. For example, looking at the first two terms of the output, we have

$$M(t,1) = \frac{1-t-\sqrt{-3t^2-2t+1}}{2t^2}, \qquad \text{and}$$
$$M_q(t,1) = \frac{\left(1-t-\sqrt{-3t^2-2t+1}\right)^2}{4t^2(-3t^2-2t+1)}$$

The Maclaurin Series of $M(t,1)$ is

$$1+t+2t^2+4t^3+9t^4+21t^5+51t^6+127t^7+323t^8+835t^9+2188t^{10}+5798t^{11}+O(t^{12}),$$

and it is the weight enumerator of the number of Motzkin paths of length n, which is A001006 on OEIS. The Maclaurin series of $M_q(t,1)$ is

$$t^2+4t^3+16t^4+56t^5+190t^6+624t^7+2014t^8+6412t^9+20219t^{10}+63284t^{11}+O(t^{12}),$$

which is the weight enumerator of the total area under all Motzkin paths of length n and A057585 on OEIS.

We also get higher factorial moments. For example,
$M_{qq}(t,1) = 1/2(6(-3t^2-2t+1)^{1/2}t^2+9t^2-(-3t^2-2t+1)^{1/2}t+6t+3(-3t^2-2t+1)^{1/2}-3)(-1+t+(-3t^2-2t+1)^{1/2})/(3t^2+2t-1)^3$,

and

$M_{qqq}(t,1) = -3/2(9(-3t^2-2t+1)^{1/2}t^4-9t^5+18(-3t^2-2t+1)^{1/2}t^3+51t^4-23(-3t^2-$

$2t+1)^{1/2}t^2-19t^3+4(-3t^2-2t+1)^{1/2}t+29t^2-4(-3t^2-2t+1)^{1/2}-8t+4)(t-1+(-3t^2-2t+1)^{1/2})/(3t^2+2t-1)^4$.

The Maclaurin series of $M_{qq}(t,1)$ is

$$2t^3+24t^4+142t^5+720t^6+3224t^7+13478t^8+53508t^9+204698t^{10}+O(t^{11}),$$

and the Maclaurin series of $M_{qqq}(t,1)$ is

$$30t^4+336t^5+2742t^6+17268t^7+95388t^8+477900t^9+2235876t^{10}+O(t^{11}).$$

The weight enumerator for the sum of the squares of the areas of Motzkin paths of length n is given by the Maclaurin series of $M_{qq}(t,1)+M_q(t,1)$,

$$t^2+6t^3+40t^4+198t^5+910t^6+3848t^7+15492t^8+59920t^9+224917t^{10}+O(t^{11}),$$

and the weight enumerator for the sum of the cubes of the areas of Motzkin paths of length n is given by the Maclaurin series of $M_{qqq}(t,1)+3M_{qq}(t,1)+M_q(t,1)$,

$$t^2+10t^3+118t^4+818t^5+5092t^6+27564t^7+137836t^8+644836t^9+2870189t^{10}+O(t^{11}).$$

None of these appear on OEIS as of September 12, 2023.

Dyck Paths

Looking at Dyck paths, we input

DerK(1,0,t^2*q,q,t,10,f).

The first three terms of the output gives

$$
\begin{aligned}
D(t,1) &= \frac{1-\sqrt{1-4t^2}}{2t^2}\\
D_q(t,1) &= \frac{(1-\sqrt{1-4t^2})^2}{16t^4-4t^2}\\
D_{qq}(t,1) &= \frac{(8t^2\sqrt{1-4t^2}+12t^2+3\sqrt{1-4t^2}-3)(-1+\sqrt{1-4t^2})}{2}
\end{aligned}
$$

The Maclaurin series of $D(t,1)$ is

$$1+t^2+2t^4+5t^6+14t^8+42t^{10}+132t^{12}+429t^{14}+1430t^{16}+O(t^{18}),$$

which is the weight enumerator of all Dyck paths of length n and A000108 on OEIS [22], https://oeis.org/A000108. The Maclaurin series of $D_q(t,1)$ is

$$t^2+6t^4+29t^6+130t^8+562t^{10}+2380t^{12}+9949t^{14}+41226t^{16}+O(t^{18}),$$

the weight enumerator for the total area of all Dyck paths of length n, which is A008549 on OEIS [22], https://oeis.org/A008549.

The Maclaurin series of $D_{qq}(t,1)$ is

$$14t^4+160t^6+1226t^8+7864t^{10}+45564t^{12}+247136t^{14}+1279810t^{16}+6404424t^{18}+O(t^{20}),$$

and the Maclaurin series of $D_{qqq}(t,1)$ is

$$24t^4+840t^6+11736t^8+114744t^{10}+922224t^{12}+6541776t^{14}+42543480t^{16}+O(t^{18}).$$

The weight enumerator for the sum of the squares of the areas of Dyck paths of length n is given by the Maclaurin series of $D_{qq}(t,1) + D_q(t,1)$,

$$t^2 + 20t^4 + 189t^6 + 1356t^8 + 8426t^{10} + 47944t^{12} + 257085t^{14} + 1321036t^{16} + O(t^{18}).$$

The weight enumerator for the sum of the cubes of the areas of Dyck paths of length n is given by the Maclaurin series of $D_{qqq}(t,1) + 3D_{qq}(t,1) + D_q(t,1)$,

$$t^2+72t^4+1349t^6+15544t^8+138898t^{10}+1061296t^{12}+7293133t^{14}+46424136t^{16}+O(t^{18}).$$

None of these appear on OEIS as of September 12, 2023.

Further Study

For further study, we can look at the average areas and the variance. Given a family of paths, let $a_0(n)$ be the number of such paths of length n, $a_1(n)$ be the total area under such paths of length n, and $a_2(n)$ be the sum of the squares of the areas under such paths of length n. Using the accompanying Maple package `qEW.txt`, we can generate 10,000 (or more) terms of the sequences of the average areas $\left\{\dfrac{a_1(n)}{a_0(n)}\right\}$ and the variances $\left\{\dfrac{a_2(n)}{a_0(n)} - \left(\dfrac{a_1(n)}{a_0(n)}\right)^2\right\}$ and use numerics for the asymptotics.

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