## Solutions to Dr. Z.'s Math 354 REAL Quiz #9

**1.** Consider the following transportation problem, where **s** is the **supply vector**, **d** is the **demand vector**, and **C** is the **cost matrix** between the supply sites and the demand sites.

$$\mathbf{C} = \begin{bmatrix} 5 & 10 & 3 \\ 10 & 7 & 4 \\ 5 & 5 & 5 \end{bmatrix} \quad , \quad \mathbf{s} = \begin{bmatrix} 162 \\ 166 \\ 29 \end{bmatrix} \quad , \quad \mathbf{d} = \begin{bmatrix} 153 \\ 193 \\ 11 \end{bmatrix} \quad .$$

**(a)** (1 point) Explain why

$$\begin{bmatrix} 151 & 0 & 11 \\ 0 & 166 & 0 \\ 2 & 27 & 0 \end{bmatrix}$$

is a **basic fesible solution**.

**(b)**: ( 7 points) Perform **one** iteration in the transportation algorithm to get a cheaper solution, or prove that none exists (i.e. that the above solution is optimal).

**Sol. to 1(a)**: The entries of the solutions are all **non-negative** (i.e. 0 or positive),

The sum of the first row, $151 + 0 + 11 = 162$ equals the first component of $s$

The sum of the second row, $0 + 166 + 0 = 166$ equals the second component of $s$

The sum of the third row, $2 + 27 + 0 = 29$ equals the third componet of $s$

The sum of the first column, $151 + 0 + 2 = 153$ equals the first component of $d$

The sum of the second column, $0 + 166 + 27 = 193$ equals the second component of $d$

The sum of the third column, $11 + 0 + 0 = 11$ equals the third component of $d$.

**Sol. of 1(b)**: The set of **basic variables** is

$$\textbf{Basic} = \{x_{11}, x_{13}, x_{22}, x_{31}, x_{32}\} \quad .$$

The set of **non-basic variables** is

$$\textbf{NonBasic} = \{x_{12}, x_{21}, x_{23}, x_{33}\} \quad .$$

The **dual variables** are $v_1, v_2, v_3, w_1, w_2, w_3$, and the set of equations (in the dual problem) that we have to solve (recall that each basic variable $x_{ij}$ yields an equation $v_i + w_j = c_{ij}$) is

$$x_{11} : v_1 + w_1 = 5 \quad ,$$

$$x_{13} : v_1 + w_3 = 3 \quad ,$$

$$x_{22} : v_2 + w_2 = 7 \quad ,$$

$$x_{31} : v_3 + w_1 = 5 \quad ,$$

$$x_{32} : v_3 + w_2 = 5 \quad .$$

Setting $v_1 = 0$ and plugging-in we get the following solution

$$v_1 = 0 \quad , \quad v_2 = 2 \quad , \quad v_3 = 0 \quad ,$$

$$w_1 = 5 \quad , \quad w_2 = 5 \quad , \quad w_3 = 3 \quad .$$

Plugging them into the Non-Basic cells, we get

$$x_{12} : v_1 + w_2 - c_{12} = 0 + 5 - 10 = -5 \quad ,$$

$$x_{21} : v_2 + w_1 - c_{21} = 2 + 5 - 10 = -3 \quad ,$$

$$x_{23} : v_2 + w_3 - c_{23} = 2 + 3 - 4 = 1 \quad ,$$

$$x_{33} : v_3 + w_3 - c_{33} = 0 + 3 - 5 = -2 \quad .$$

The largest value (and in this case, the only positive one) is for $x_{23}$, so the **departing variable** is $x_{23}$, and we have to find an **alternating horizontal-vertical** path from cell $[2, 3]$ back to it only visiting basic cells (i.e. cells with currently positive values).

The only such path is the following path of length 6 (or its reverse, i.e. the same path travelled in the opposite direction)

$$[2, 3] \to [2, 2] \to [3, 2] \to [3, 1] \to [1, 1] \to [1, 3] \to [2, 3] \quad .$$

The even-indexed locations (variables) are $x_{22} = 166$, $x_{31} = 2$, $x_{13} = 11$. The **smallest** value is 2 and it is at cell $[3, 1]$. We update the cells along the above path by subtracting 2 from the even-indexed cells and adding 2 to the odd-indexed cells. Hence

$$x_{23} \leftarrow x_{23} + 2 = 0 + 2 = 2 \quad ,$$

$$x_{22} \leftarrow x_{22} - 2 = 166 - 2 = 164 \quad ,$$

$$x_{32} \leftarrow x_{32} + 2 = 27 + 2 = 29 \quad ,$$

$$x_{31} \leftarrow x_{31} - 2 = 2 - 2 = 0 \quad ,$$

$$x_{11} \leftarrow x_{11} + 2 = 151 + 2 = 153 \quad ,$$

$$x_{13} \leftarrow x_{13} - 2 = 11 - 2 = 9 \quad ,$$

The other cells (variables),namely $x_{21}$, $x_{21}$ and $x_{33}$ remain the same. The outcome of the first iteration, hence is.

**Ans. to 1(b)**:

$$\begin{bmatrix} 153 & 0 & 9 \\ 0 & 164 & 2 \\ 0 & 29 & 0 \end{bmatrix} \quad .$$