Solutions to Dr. Z.'s Math 354 REAL Quiz #6

1. (8 pts.) Solve the following linear programming problem, using the M-method. (No credit for using the graphical, or any other method). Explain everything.

Maximize x, subject to the restrictions

$$x + y = 5 \quad , \quad x - y = 1 \quad , \quad x \geq 0 \quad , \quad y \geq 0 \quad .$$

Sol. to 1: Since neither x nor y may be taken as basic variables, we must introduce **artificial variables**, let's call then u and v.

The goal function is written as

$$z = x - Mu - Mv \quad ,$$

where M is VERY BIG. The idea is that the only way that we can maximize the goal is that at the end u and v are forced to be 0.

We rewrite the system as

$$x + y + u = 5 \quad , \quad x - y + v = 1 \quad , \quad x \geq 0 \quad , \quad y \geq 0 \quad , \quad u \geq 0 \quad , \quad v \geq 0 \quad .$$

The very first thing that we have to do is to get rid of u and v in the expression for z.

From the two equalities, we can express the articicial variables u and v in terms of the natural variables x and y

$$u = 5 - x - y \quad , \quad v = 1 - x + y \quad .$$

Substituting into the expression for z, and simplifying, we have

$$z = x - Mu - Mv = x - M(5-x-y) - M(1-x+y) = x - 5M + Mx + My - M + Mx - My = -6M + (2M+1)x \quad .$$

Moving, as usual, everything, except for the pure number (in this case $-6M$) to the left side, we get

$$-(2M+1)x + 0 \cdot y + 0 \cdot y + 0 \cdot u + 0 \cdot v + z \;=\; -6M \quad .$$

Now u and v are basic variables (u only shows up in the first equation, and has coefficient 1, and v only shows up in the second equation, and has coefficient 1. Of course neither of them shows up in the goal equation.

The **initial tableau** is

	x	y	u	v	z	
u	1	1	1	0	0	5
v	1	-1	0	1	0	1
	$-(2M+1)$	0	0	0	1	$-6M$

.

The **most negative** entry in the bottom (goal) row is the $-(2M+1)$ at the first column (in this case it is the only one), so the **column entry** is the column corresponding to the variable x, so the **entering variable** is x.

To decide on the **departing variable**, we take $\theta-$ ratios.

- The θ-ratio for the first row (belonging to the basic variable u) is $\frac{5}{1} = 5$

- The θ-ratio for the second row (belonging to the basic variable v) is $\frac{1}{1} = 1$

Since 1 is the smallest, the **departing variable** is v, and the **pivot entry** is the $(2,1)$ entry.

Luckily it is already 1, so we don't need to do a scaling operation in order to make it 1. Let's indicate the entering and departing variables by arrows.

$$\begin{array}{c|ccccc|c} & x^{\downarrow} & y & u & v & z & \\ & & & & & & \\ u & 1 & 1 & 1 & 0 & 0 & 5 \\ ^{\leftarrow}v & 1 & -1 & 0 & 1 & 0 & 1 \\ & & & & & & \\ & -(2M+1) & 0 & 0 & 0 & 1 & -6M \end{array}.$$

Since the entering basic variable is x, everything in its column, except for the pivot at entry $(2,1)$, must become 0.

To that end, we perform the row operations $r_1 - r_2 \to r_1$ and $r_3 + (2M+1)r_2 \to r_3$, getting the tableau

$$\begin{array}{c|ccccc|c} & x & y & u & v & z & \\ & & & & & & \\ u & 0 & 2 & 1 & -1 & 0 & 4 \\ x & 1 & -1 & 0 & 1 & 0 & 1 \\ & & & & & & \\ & 0 & -(2M+1) & 0 & 2M+1 & 1 & -4M+1 \end{array}.$$

Now the most negative entry of the last (goal) row is at the y column, so the **entering variable** is y.

To decide on the **departing variable**, we take $\theta-$ ratios.

- The θ-ratio for the first row (belonging to the basic variable u) is $\frac{4}{2} = 2$

- The θ-ratio for the second row (belonging to the basic variable x) is negative, so we disregard it.

This means that the **departing variable** is u, and the **pivot** entry is the $(1,2)$. Let's make it 1 by performing $\frac{1}{2}r_1 \to r_1$, and at the same time indicate the entering and departing variables by arrows.

$$\begin{array}{c|ccccc|c} & x & y^{\downarrow} & u & v & z & \\ & & & & & & \\ {}^{\leftarrow}u & 0 & 1 & 1/2 & -1/2 & 0 & 2 \\ x & 1 & -1 & 0 & 1 & 0 & 1 \\ & & & & & & \\ & 0 & -(2M+1) & 0 & 2M+1 & 1 & -4M+1 \end{array} .$$

Since y must become a basic variable everything in its column, except for the pivot at entry $(1,2)$, must become 0.

To that end, we perform the row operations $r_2 + r_1 \to r_2$ and $r_3 + (2M+1)r_1 \to r_3$. At the same time we replace the departing basic variable u by the entering basic variable y. We get the new tableau.

$$\begin{array}{c|ccccc|c} & x & y & u & v & z & \\ & & & & & & \\ y & 0 & 1 & 1/2 & -1/2 & 0 & 2 \\ x & 1 & 0 & 1/2 & 1/2 & 0 & 3 \\ & & & & & & \\ & 0 & 0 & M+1/2 & M+1/2 & 1 & 3 \end{array} .$$

Since there are no longer any negative entries in the last (goal) row, this is the **final tableau**. and we get that optimal solution is $y = 2$, $x = 3$ (and $u = 0$ and $v = 0$ as they should). The **optimal value** is 3.

Ans. to 1: The optimal solution is $x = 3$ and $y = 2$ and the optimal value is $z = 3$.

Comment: For this problem it is **VERY STUPID** to use the M-method, or for that matter, linear programming. High school algebra suffices. The only solution of the system of two equations and two unknowns

$$x + y = 5 \quad , \quad x - y = 1$$

is $x = 3$ and $y = 2$, and since neither of them is negative, it is a feasible solution. So the set of feasible solution consists of only one member,$(x, y) = (3, 2)$, and hence there is not much choice and the maximum (and for that matter, also the minimum) of the goal function is at that same point, and its value is 3. The point of this quiz is to test your familiarity with the M method. Any realistic problem would take longer than ten minutes to do (by hand.)