Solutions to Dr. Z.'s Math 354 REAL Quiz #3

Corrected thanks to Keonho Roh

1. (3 pts.) Write the following linear programming problem in matrix form. First introduce slack variables, and give them names.

Maximize $w = 2x + 3y + 5z$ subject to

$$x + 5y + z \geq 5 \quad , \quad 4x + 2y + 5z \geq 12 \quad , \quad x + y + z = 10 \quad , \quad x \geq 0 \quad , \quad y \geq 0 \quad , \quad z \geq 0 \quad .$$

Sol. to 1: **First** we have to convert the constraints to **standard form**

Maximize $w = 2x + 3y + 5z$ subject to

$$-x - 5y - z \leq -5 \quad , \quad -4x - 2y - 5z \leq -12 \quad , \quad x + y + z = 10 \quad , \quad x \geq 0 \quad , \quad y \geq 0 \quad , \quad z \geq 0 \quad .$$

(To convert $\geq$ to $\leq$ we multiply both sides by -1

There are two inequalities (we can leave the equality alone), so we need two slack-variables, let's call them u and v. We have, in everyday notation

$$-x - 5y - z + u = -5 \quad , \quad -4x - 2y - 5z + v = -12 \quad , \quad x + y + z = 10 \quad ,$$

$$x \geq 0 \quad , \quad y \geq 0 \quad , \quad z \geq 0 \quad , \quad u \geq 0 \quad , \quad v \geq 0 \quad .$$

In **matrix notation** this is:

$$\text{Maximize } z = \begin{bmatrix} 2 & 3 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ y \\ u \\ v \end{bmatrix}$$

subject to

$$\begin{bmatrix} -1 & -5 & -1 & 1 & 0 \\ -4 & -2 & -5 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 10 \end{bmatrix} \quad ,$$

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad .$$

2. (5 pts.) Sketch the set of feasible solutions to the given linear programming problem

Maximize $z = 2x + 3y$ subject to:

$$x + y \leq 4 \quad , \quad 3x + y \leq 6 \quad , \quad x + 3y \leq 6 \quad ,$$

$$x \geq 0 \quad , \quad y \geq 0 \qquad k = 6, 12 \quad .$$

Comment: Since I cancelled the original part (b) (the k part), the goal function is not important for this question, only plotting the feasible region.

The region is the part of the **positive quadrant** $\{(x, y)|x \geq 0, y \geq 0\}$ that is **below** the line $3x + y = 6$ and **below** the line $x + 3y = 6$ and also under line the $x + y = 4$. But the common region below both the lines $3x + y = 6$ and the line $x + 3y = 6$ is automatically below the line $x + y = 4$, so we can forget about it.

The point of intersection of $3x + y = 6$ and $x + 3y = 6$ is $(\frac{3}{2}, \frac{3}{2})$, so the feasible region is the quadrilateral with vertices (alias *extreme points*)

$$(0, 0), (2, 0), (\frac{3}{2}, \frac{3}{2}), (0, 2) \quad .$$

Comment: By now you know that to find the maximum you **plug-in** the above extreme points and see who is the "winner" (or in case of ties, "winners"). This was **not** required in this quiz, since this is the material of section 1.4 (that will be covered in the next quiz).

Just for fun here it is:

- when $x = 0, y = 0$ $z = 2 \cdot 0 + 3 \cdot 0 = 0$
- when $x = 2, y = 0$ $z = 2 \cdot 2 + 3 \cdot 0 = 4$
- when $x = \frac{3}{2}, y = \frac{3}{2}$ $z = 2 \cdot \frac{3}{2} + 3 \cdot \frac{3}{2} = \frac{15}{2}$
- when $x = 0, y = 2$ $z = 2 \cdot 0 + 3 \cdot 2 = 6$

Since $\frac{15}{2}$ is the the largest value, the answer is: The maximum is $\frac{15}{2}$ and it takes place when $x = \frac{3}{2}, y = \frac{3}{2}$.