Solutions to Dr. Z.'s Math 354 REAL Quiz #10

1. Solve the following assignment problem with 5 employees and 5 jobs.

$$\begin{bmatrix} 1 & 3 & 6 & 3 & 5 \\ 2 & 3 & 6 & 7 & 3 \\ 4 & 7 & 8 & 7 & 4 \\ 3 & 2 & 3 & 7 & 3 \\ 5 & 2 & 4 & 8 & 2 \end{bmatrix} \quad ,$$

Sol. to 1: We find an **equivalent** problem (i.e. a problem with the **same solution**) where every row has at least one zero, by subtracting from each row its smallest entry.

Doing

$$r_1 - 1 \to r_1 \quad , \quad r_2 - 2 \to r_2 \quad , \quad r_3 - 4 \to r_3 \quad , \quad r_4 - 2 \to r_4 \quad , \quad r_5 - 2 \to r_5 \quad ,$$

gives the new problem

$$\begin{bmatrix} 0 & 2 & 5 & 2 & 4 \\ 0 & 1 & 4 & 5 & 1 \\ 0 & 3 & 4 & 3 & 0 \\ 1 & 0 & 1 & 5 & 1 \\ 3 & 0 & 2 & 6 & 0 \end{bmatrix} \quad ,$$

Next we have to make sure that every column has at least one zero, by subtracting from each column that has no zeros, its smallest entry, like we did with the rows. The only columns that still has lingering zeros are Column 3, and Column 4. Doing

$$c_3 - 1 \to c_3 \quad , \quad c_4 - 2 \to c_4 \quad ,$$

we get the simpler equivalent problem

$$\begin{bmatrix} 0 & 2 & 4 & 0 & 4 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 3 & 3 & 1 & 0 \\ 1 & 0 & 0 & 3 & 1 \\ 3 & 0 & 1 & 4 & 0 \end{bmatrix} \quad ,$$

The next step is to do try and do **match-making**. It is easy to find a perfect matching **by inspection** (but you are welcome to use the official *alternating paths* algorithm).

• Since Row 2 is only willing to marry Column 1, we must star the zero at entry $(2, 1)$.

• Since Column 3 is only willing to marry Row 4, we must star the zero at entry $(4, 3)$.

Now the zero at entry $(4, 2)$ can't be starred, wich leaves only one zero in Column 2, namely the zero at entry $(5, 2)$, that must be starred, and now it is really easy to do the match-making, and we get

$$\begin{bmatrix} 0 & 2 & 4 & 0^* & 4 \\ 0^* & 1 & 3 & 3 & 1 \\ 0 & 3 & 3 & 1 & 0^* \\ 1 & 0 & 0^* & 3 & 1 \\ 3 & 0^* & 1 & 4 & 0 \end{bmatrix} \quad ,$$

Converting the 0^* to 1 and all the other entries to 0, we get the **permutation matrix**, that is a solution to our assignment problem.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad .$$

This corresponds to the permutation in **two-line notation**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 3 & 2 \end{pmatrix} \quad ,$$

and in **one-line notation** 41532.

Ans. to 1: 41532 (i.e. employee 1 will get job 4 etc.)