Solutions to Attendance Quiz for Lecture 9

Corrected version (Thanks to Amber Rawson, who won a dollar.)

[In the final tableau the RHS entry in the v column was previously stated as 4 rather than the correct 2, and this error was carried on. Since v is a slack variable, it did not change the final answer.]

1. Solve the following linear programming problem using the simplex method.

Maximize $x + 4y$ subject to the restrictions

$$2x + 3y \le 6 \quad , \quad 3x + 2y \le 6 \quad , \quad 5x + 5y \le 11 \quad ,$$

$$x \ge 0 \quad , \quad y \ge 0 \quad .$$

Sol. to 1: We first convert to **canonical form** by introducing the **slack variables** u, v, w.

The problem now is

Maximize $z = x + 4y$ subject to the restrictions

$$2x + 3y + u \;=\; 6 \quad , \quad 3x + 2y + v \;=\; 6 \quad , \quad 5x + 5y + w \;=\; 11 \quad ,$$

$$x \ge 0 \quad , \quad y \ge 0 \quad , \quad u \ge 0 \quad , \quad v \ge 0 \quad , \quad w \ge 0 \quad .$$

Writing $z = x + 4y$ as $-x - 4y + z = 0$, the *initial tableau* is

$$\begin{array}{c|cccccc|c} & x & y & u & v & w & z & \\ - & - & - & - & - & - & - & - \\ u & 2 & 3 & 1 & 0 & 0 & 0 & 6 \\ v & 3 & 2 & 0 & 1 & 0 & 0 & 6 \\ w & 5 & 5 & 0 & 0 & 1 & 0 & 11 \\ - & - & - & - & - & - & - & - \\ & -1 & -4 & 0 & 0 & 0 & 1 & 0 \end{array} \quad .$$

Since the **most negative** entry in the last (goal) row is at the second column, the second column is the **pivot column**. The **θ-ratios** are

- $6/3 = 2$ for the first row

- $6/2 = 3$ for the second row

- $11/5 = 2.1$ for the third row

Since the smallest of these numbers (that happens to be 2) occurs at the first row, the **pivot** entry is the 3 at the intersection of the first row and second column. It follows that the **departing**

(basic) variable is u and the **entering (basic) variable** is y. Let's indicate this as follows, and at the same time divide the first row by 3 in order to make the pivot 1.

$$\begin{array}{c|cccccc|c}
 & x & \mathbf{y}^{\downarrow} & u & v & w & z & \\
- & - & - & - & - & - & - & - \\
{}^{\leftarrow}\mathbf{u} & 2/3 & 1 & 1/3 & 0 & 0 & 0 & 2 \\
v & 3 & 2 & 0 & 1 & 0 & 0 & 6 \\
w & 5 & 5 & 0 & 0 & 1 & 0 & 11 \\
- & - & - & - & - & - & - & - \\
 & -1 & -4 & 0 & 0 & 0 & 1 & 0
\end{array} \quad .$$

But in order to make y a basic variable, everything in its column, except for the pivot entry, must be 0. So we perform **elementary row operations**.

- To make the $(2,2)$ entry 0 we perform $r_2 - 2r_1 \to r_2$,
- To make the $(3,2)$ entry 0 we perform $r_3 - 5r_1 \to r_3$,
- To make the $(4,2)$ entry 0 (in the last, goal, row) we perform $r_4 + 4r_1 \to r_4$.

Performing these three elementary row operations, and replacing the departing variable u in the leftmost column by the entering variable y, we get the following new tableau.

$$\begin{array}{c|cccccc|c}
 & x & y & u & v & w & z & \\
- & - & - & - & - & - & - & - \\
y & 2/3 & 1 & 1/3 & 0 & 0 & 0 & 2 \\
v & 5/3 & 0 & -2/3 & 1 & 0 & 0 & 2 \\
w & 5/3 & 0 & -5/3 & 0 & 1 & 0 & 1 \\
- & - & - & - & - & - & - & - \\
 & 5/3 & 0 & 4/3 & 0 & 0 & 1 & 8
\end{array} \quad .$$

We lucked out! After only one step we arrived at the solution (usually it takes more than one step), since none of the entries in the last row (the goal row) are negative.

Hence an optimal solution is $y = 2$, $v = 2$ and $w = 1$, and since the **non-basic variables** are x and u, we have that $x = 0$ and $u = 0$. Hence an optimal solution is

$$(x, y, u, v, w) = (0, 2, 0, 2, 1) \quad .$$

But we do not care about the slack variables, so the **truncated solution** is

$$(x, y) = (0, 2) \quad ,$$

and the optimal value is $0 + 4 \cdot 2 = 8$ that is also seen frm the rightmost entry of the last row.

Ans. to 1: The optimal solution is $x = 0$, $y = 2$, and the optimal value is 8.