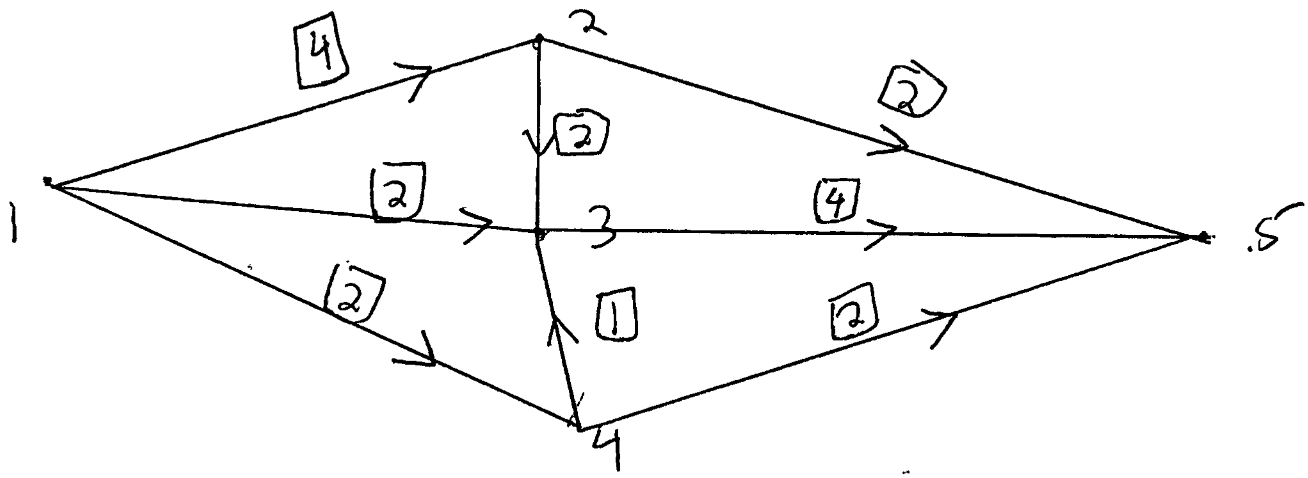
Solutions to Attendance Quiz for Lecture 22

1. In the following network vertex 1 is the source and vertex 5 is the sink (terminal). The capacities are given by the above matrix ($c_{ij} = 0$ means that there is no edge between vertex i and vertex j)

$$\begin{bmatrix} 0 & 4 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

(a) Draw the network with the source 1 on the left, the sink (terminal) 5 on the right and vertices 2, 3, and 4 in the middle. Indicate the capacities next to each edge.



(b) Find a maximal flow and state its value .

Starting with the trivial zero flow, We keep looking for **augmenting paths** joining the **source** (vertex 0) to the **sink** (vertex 5), and add the appropriate flow. We do it until there are no more augmenting path, which means that we have found the maximum flow.

An augmenting path may have two kinds of edges. a **forward edge** where the current flow is less than the capacity, or a **backward edge** where the current flow is **positive**. Since backward edges are more tricky, we first try to only use forward edges, and only look for backward edges if we are desperate.

We start with the **zero** flow

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

At the beginning there are quite a few augmenting paths. One such path is

$$1 \to 2 \to 5$$

The currently still-available flow in edge $1 \to 2$ is $4 - 0 = 4$ and the currently still-available flow in edge $2 \to 5$ is $2 - 0 = 2$. The smallest of 2 and 4 is 2 so we add a flow of 2 into these two edges. This means that we add 2 to the [1, 2] and [2, 5] entries of the flow-matrix. The current flow is:

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Since there is still some available flow in edge $1 \to 2$, as well as in edges $2 \to 3$ and $3 \to 5$, an augmented path from the source 1, to the sink 5 is:

$$1 \to 2 \to 3 \to 5$$

The currently still-available flow in edge $1 \to 2$ is $4-2 = 2$, the currently still-available flow in edge $2 \to 3$ is $2 - 0 = 2$, while the currently still-available flow in edge $3 \to 5$ is $4 - 0 = 4$. The smallest is 2 so we add 2 to the [1, 2], [2, 3], and [3, 5] entries of the above flow matrix. We now have

$$\begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Moving right along, edge $1 \to 2$ is currently **saturated**, so it can't be used anymore. But edges $1 \to 3$ and $1 \to 4$ still have some flow to spare. Since also edge $3 \to 5$ has flow to spare, a current legal augmenting path is

$$1 \to 3 \to 5$$

The currently still-available flow in edge $1 \to 3$ is $2 - 0 = 2$, the currently still-available flow in edge $3 \to 5$ is $2 - 0 = 2$, The smallest (in this case, they happen to be equal) is 2 so we add 2 to the [1, 3] and [3, 5] entries, getting the better flow matrix

$$\begin{bmatrix} 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Right now, the only unsaturated edge out of the source, 1, is $1 \to 4$. Here is a new augmented path

$$1 \to 4 \to 5$$

The currently still-available flow in edge $1 \to 4$ is $2 - 0 = 2$, the currently still-available flow in edge $4 \to 5$ is $2 - 0 = 2$. The smallest is 2, so we add 2 to the [1,4] and [4,5] entries of the flow matrix, getting

$$\begin{bmatrix} 0 & 4 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Are there still augmenting paths? No! All the edges coming out of the source, 1, are currently saturated (i.e. *fully booked*), so we have reached our destination, and the above flow matrix represents the **maximal flow**. Its value is $0 + 4 + 2 + 2 = 8$, alias $0 + 2 + 4 + 2 + 0 = 8$.

Ans. to (b): The maximal flow is

$$\begin{bmatrix} 0 & 4 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

and its value is 8.

(c): Find a minimal cut and state its value.

Sol. to (c): We look at the **connected component** of the network reachable from the source via (partial) augmenting paths. In this simple problem only 1 (the source) is reachable, and a **minimal cut** is the set of edges leading from 1 to the rest of the network. The set of these edges is

$$\{1 \to 2 \quad , \quad 1 \to 3 \quad , \quad 1 \to 4\} \quad ,$$

and the sum of the capacities is $c_{12} + c_{13} + c_{14} = 4 + 2 + 2 = 8$. Indeed this is the minimal cut, and its value is 8, confirming the famous MaxFlow=MinCut Theorem.

Ans. to (c): A minimal cut is $\{12\, , \, 13\, , \, 14\}$ and its value is 8.

Comments:

1. In this problem, we lucked out, and none of the augmenting paths needed a backward edge. In general you do need them. In that case, we look at the current value of the flow, and take the minimum of the still-available flows in the forward edges and the current (positive) flow in the backward edges, call that number α, and **add** α to the entries corresponding to the forward edges (as we did above), and **subtract** α from the entries corresponding to the backward edges.

2. In the above problem, the set of vertices reachable from the source via partial augmenting paths was the singleton set consisting of the source alone, so it was easy to get the minimal cut, simply all the edges coming out of it. In bigger problem the set of reachable vertices could be much larger, and the minimal cut is the set of all edges from a vertex in the source side to a vertex in the other side (the sink side).