Solution to Attendance Quiz for Lecture 16

1. Perform one iteration in solving the following transportation problem, where **s** is the **supply vector**, **d** is the **demand vector**, and **C** is the **cost matrix** between the supply sites and the demand sites.

$$\mathbf{C} = \begin{bmatrix} 5 & 2 & 3 & 6 \\ 2 & 7 & 7 & 4 \\ 1 & 3 & 6 & 9 \end{bmatrix} \quad , \quad \mathbf{s} = \begin{bmatrix} 100 \\ 80 \\ 140 \end{bmatrix} \quad , \quad \mathbf{d} = \begin{bmatrix} 60 \\ 60 \\ 80 \\ 120 \end{bmatrix} \quad .$$

starting from the following basic feasible solution obtained by Vogel's method (last time)

$$\begin{bmatrix} 0 & 20 & 80 & 0 \\ 0 & 0 & 0 & 80 \\ 60 & 40 & 0 & 40 \end{bmatrix}$$

Sol. 1: The set of **basic variables**, corresponding to the **positive** entries in the above matrix is

$$Basic = \{x_{12}, x_{13}, x_{24}, x_{31}, x_{32}, x_{34}\} \quad ,$$

the set of **non-basic variables** correspond to the entries that are **zero**.

$$NonBasic = \{x_{11}, x_{14}, x_{21}, x_{22}, x_{23}, x_{33}\} \quad .$$

We introduce $3+4=7$ variables $v_1, v_2, v_3, w_1, w_2, w_3, w_4$, and have to solve the system $v_i + w_j = c_{ij}$ for each basic variable x_{ij}. There are six equations:

$$x_{12} : v_1 + w_2 = 2 \quad , \quad x_{13} : v_1 + w_3 = 3 \quad , \quad x_{24} : v_2 + w_4 = 4 \quad ,$$

$$x_{31} : v_3 + w_1 = 1 \quad , \quad x_{32} : v_3 + w_2 = 3 \quad , \quad x_{34} : v_3 + w_4 = 9 \quad .$$

Since there are six equations and seven variables, one of these variables is set to 0. Since v_3 shows up in three of them it is easiest to make $v_3 = 0$. After that it is easy to get the solution. It is

$$v_1 = -1 \quad , \quad v_2 = -5 \quad , \quad v_3 = 0$$

$$w_1 = 1 \quad , \quad w_2 = 3 \quad , \quad w_3 = 4 \quad , \quad w_4 = 9 \quad .$$

In order to find the **entering variable** we must compute $v_i + w_j - c_{ij}$ for all non-basic variables.

$$x_{11} : v_1 + w_1 - c_{11} = -1 + 1 - 5 = -5 \quad , \quad x_{14} : v_1 + w_4 - c_{14} = -1 + 9 - 6 = 2 \quad ,$$

$$x_{21} : v_2 + w_1 - c_{21} = -5 + 1 - 2 = -6 \quad , \quad x_{22} : v_2 + w_2 - c_{22} = -5 + 3 - 7 = -9 \quad ,$$

$$x_{23} : v_2 + w_3 - c_{23} = -5 + 4 - 7 = -8 \quad , \quad x_{33} : v_3 + w_3 - c_{33} = 0 + 4 - 6 = -2 \quad .$$

The **largest** value corresponds to x_{14}, and this is the **entering variable**.

We need to find an alternating horizontal-vertical path starting at cell $[1,4]$ and traveling via basic cells, returning back to $[1,4]$. The only such path is

$$[1,4] \to [1,2] \to [3,2] \to [3,4] \to [1,4] \quad .$$

The even locations are $[1,2]$ and $[3,4]$. We have, currently $x_{12} = 20$ and $x_{34} = 40$. The smaller value is 20, hence x_{12} is the **departing variable**, and we update

$$x_{14} = 0 + 20 = 20 \quad , \quad x_{12} = 20 - 20 = 0 \quad , \quad x_{32} = 40 + 20 = 60 \quad , \quad x_{34} = 40 - 20 = 20 \quad .$$

All the other values of x_{ij} stay the same. The new feasible solution is

$$\begin{bmatrix} 0 & 0 & 80 & 20 \\ 0 & 0 & 0 & 80 \\ 60 & 60 & 0 & 20 \end{bmatrix}$$

Its cost is

$$80 \cdot 3 + 20 \cdot 6 + 80 \cdot 4 + 60 \cdot 1 + 60 \cdot 3 + 20 \cdot 9 = 1100 \quad .$$

Recall that the cost of the initial feasible solution was 1140.

Note: If you try to apply one more iteration, you would get that all the values are non-positive, and you would realize that this is the final solution. Please do it!