Solutions to Attendance Quiz for Lecture 13

1. For the following linear programming problem

Minimize $z = 4x + 6y$ subject to

$$x + 3y \geq 5 \quad , \quad 2x + y \geq 4 \quad , \quad x \geq 0 \quad , \quad y \geq 0 \quad .$$

(i) Solve it (using any method), not forgetting to find the **optimal value**.

(ii) Set-up the **dual problem** .

(iii) Solve the dual problem (using any method), not forgetting to find the **optimal value**.

(iv) If you solved them both correctly, the optimal values for (i) and (iii) are the same. Is this a coincidence? If not what theorem does it follow from?

Sol. to 1(i): Since we only have two variables. The easiest way is the **graphical method**. The feasible region is the region in the first quadrant of the plane **above** the lines $x + 3y = 5$ and $2x + y = 4$. This is an infinite region with extreme points $\{(5,0), (0,4), (\frac{7}{5}, \frac{6}{5}\}$

(The last point is the point of intersection of the two lines obtained by solving the system of two equations and two unknowns $\{x + 3y = 5, 2x + y = 4\}$ (you do it!)

Plugging in into the goal function, we get

$z(5,0) = 4 \cdot 5 + 6 \cdot 0 = 20$

$z(0,4) = 0 \cdot 5 + 6 \cdot 4 = 24$

$z(\frac{7}{5}, \frac{6}{5}) = 0 \cdot 5 + 6 \cdot 4 = \frac{64}{5} = 12\frac{4}{5}$

Since the minimal value is gotten when $x = \frac{7}{5}$ and $y = \frac{6}{5}$, this is the **optimal solution** (or solution for short). The **optimal value** is $\frac{64}{5} = 12\frac{4}{5}$.

Ans. to 1(i): The optimal solution is $\mathbf{x_0} = [\frac{7}{5}, \frac{6}{5}]^T$ and the optimal value $\mathbf{c}^T\mathbf{x_0}$ is $\frac{64}{5}$.

Sol. to 1(ii):

In matrix notation the **primal problem** is

Minimize $z = [4,6][{x \atop y}]$ subject to

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad , \quad \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

When we **dualize** "Minimize" becomes "Maximize" (and vice-versa) "$\geq$" becomes "$\leq$" (except for the fact that the variables must always be non-negative). the matrix of coefficients becomes its transpose, the vector featured in the goal function becomes the right hand side of the inequalities, and the vector on the right hand side of the inequalities becomes the vector featured in the goal function.

Hence, the dual problem, in matrix notation, is

Maximize $z' = [5, 4]\mathbf{w}$ subject to

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \mathbf{w} \leq \begin{bmatrix} 4 \\ 6 \end{bmatrix} \quad , \quad \mathbf{w} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

In everyday notation this is (let's use x, y again, computers do not mind if we rename).

Maximize $z' = 5x + 4y$ subject to the constraints

$$x + 2y \leq 4 \quad , \quad 3x + y \leq 6 \quad , \quad x \geq 0 \quad , \quad , y \geq 0 \quad .$$

This finishes part (ii).

Sol. to 1(iii):

Now we have to solve it. Once again, the fastest way is the graphical method.

The **feasible region** is the region in the first quadrant of the xy plane **under** the lines $x + 2y = 4$ and $3x + y = 6$. This is a quadrilateral with the set of **extreme points** $\{(0, 0), (2, 0), (\frac{8}{5}, \frac{6}{5}), (0, 2)\}$. Plugging into the goal function we get

$z(0, 0) = 5 \cdot 0 + 4 \cdot 0 = 0$

$z(2, 0) = 5 \cdot 2 + 4 \cdot 0 = 10$

$z(\frac{8}{5}, \frac{6}{5}) = 5 \cdot \frac{8}{5} + 4 \cdot \frac{6}{5} = \frac{64}{5} = 12\frac{4}{5}$.

$z(0, 2) = 5 \cdot 2 + 4 \cdot 2 = 8$

Ans. to 1(iii): The **maximal value** is when $x = \frac{8}{5}$, $y = \frac{6}{5}$, and the **optimal value** is $\frac{64}{5}$.

Sol. to 1(iv): We saw that the **optimal values** for the primal and dual problems were **the same** (namely $\frac{64}{5} = 12.8$). This is **no coincidence**. The famous **Duality Theorem** states that this is **always** true (whenever there is a solution).