Solutions to Attendance Quiz for Lecture 10

1. Set up the initial simplex tableau for solving the following linear programming problem using the big-M method.

Minimize $x_1 + x_2$ subject to the constraints

$$2x_1 + x_2 \geq 2 \quad ,$$

$$x_1 + 2x_2 \geq 2 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad .$$

Sol. of 1: We first convert the problem to **canonical form**. Instead of minimizing $x_1 + x_2$ we maximize $-x_1 - x_2$ and we introduce the two **slack variables** x_3 and x_4, to get the following Linear Programming problem in canonical form.

Maximize $z = -x_1 - x_2$ subject to the constraints

$$2x_1 + x_2 - x_3 \ = \ 2 \quad ,$$

$$x_1 + 2x_2 - x_4 \ = 2 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \geq 0 \quad , \quad x_4 \geq 0 \quad .$$

(Note that the $-x_3$ and $-x_4$ in the equations since they came from $\geq$ inequalities).

Next we introduce two **artificial variables**, y_1, y_2 and a symbol M denoting a HUGE positive number (for example a googolplex). At the end of the day, we want the values of y_1 and y_2 to be 0, so the new goal function is $z = -x_1 - x_2 - My_1 - My_2$ and the new equations are obtained by adding y_1 and y_2 to the two constraints.

Our new (but **equivalent**) optimization problem is

Maximize $z = -x_1 - x_2 - My_1 - My_2$ subject to the constraints

$$2x_1 + x_2 - x_3 + y_1 \ = \ 2 \quad ,$$

$$x_1 + 2x_2 - x_4 + y_2 \ = 2 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \geq 0 \quad , \quad x_4 \geq 0 \quad , \quad y_1 \geq 0 \quad , \quad y_2 \geq 0 \quad .$$

Our next goal in life is the rather tedious task to express z without y_1 and y_2, i.e. only in terms of the 'natural' variables, x_1, x_2, x_3, x_4. To that end, we first use the two equations to express y_1 and y_2 in terms of the natural variables x_1, x_2, x_3, x_4.

We get

$$y_1 = 2 - 2x_1 - x_2 + x_3 \quad ,$$

$$y_2 = 2 - x_1 - 2x_2 + x_4 \quad .$$

Since the **goal equation** is

$$x_1 + x_2 + My_1 + My_2 + z = 0 \quad ,$$

we get

$$x_1 + x_2$$
$$+M(2 - 2x_1 - x_2 + x_3)$$
$$+M(2 - x_1 - 2x_2 + x_4)$$
$$+z = 0 \quad .$$

Opening up parantheses, this is

$$x_1 + x_2$$
$$+2M - 2Mx_1 - Mx_2 + Mx_3$$
$$+2M - Mx_1 - 2Mx_2 + Mx_4$$
$$+z = 0 \quad .$$

Collecting terms we get the **goal equation**

$$4M + (1 - 3M)x_1 + (1 - 3M)x_2 + Mx_3 + Mx_4 + z = 0 \quad .$$

Moving the $4M$ to the right hand side, we get

$$(1 - 3M)x_1 + (1 - 3M)x_2 + Mx_3 + Mx_4 + z = -4M \quad .$$

Our problem is now the following.

Maximize z subject to

$$2x_1 + x_2 - x_3 + y_1 \ = \ 2 \quad ,$$
$$x_1 + 2x_2 - x_4 + y_2 \ = 2 \quad ,$$
$$(1 - 3M)x_1 + (1 - 3M)x_2 + Mx_3 + Mx_4 + z = -4M \quad .$$
$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , x_3 \geq 0 \quad , \quad x_4 \geq 0 \quad , \quad y_1 \geq 0 \quad , \quad y_2 \geq 0 \quad .$$

Now, and only now, we are ready to set up the **initial tableau**.

The **initial tableau** is:

$$\begin{array}{c|ccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & y_1 & y_2 & z & \\
- & - & - & - & - & - & - & - & - \\
y_1 & 2 & 1 & -1 & 0 & 1 & 0 & 0 & 2 \\
y_2 & 1 & 2 & 0 & -1 & 0 & 1 & 0 & 2 \\
- & - & - & - & - & - & - & - & - \\
 & 1-3M & 1-3M & M & M & 0 & 0 & 1 & -4M
\end{array} \quad .$$

Now we are ready to apply the simplex algorithm as usual, but keeping in mind that M is a **FIXED** big number.

WARNING: M is fixed number, it is **NOT** infinity. So it is wrong, for example, to replace $2M$ by M. In computer implementation, you can take M to be 10^{100}.