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MATH 354 ~~(3)~~, Dr. Z. , Exam 2, Mon. Nov. 27, 2023, 10:25-11:35am, TIL-246

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM

No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.

Do not write below this line

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1. 10 (out of 10)
2. 10 (out of 10)
3. 10 (out of 10)
4. 10 (out of 10)
5. 10 (out of 10)
6. 10 (out of 10)
7. 10 (out of 10)
8. 10 (out of 10)
9. 20 (out of 20)

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tot.: ~~(out of 100)~~ 100

GOOD JOB!

Reminder from Linear Algebra:

The inverse of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\frac{1}{\det(A)}[A_{ij}]^T \quad ,$$

where A_{ij} is $(-1)^{i+j}$ times the determinant of the (i, j) **minor**, which is the $(n-1)\times(n-1)$ matrix obtained by removing the i-th row and the j-th column.

Another way to find the inverse A^{-1} of an $n \times n$ matrix, A, is to stick the identity matrix I_n right after it, getting $[A|I_n]$, perform Gauss-Jordan elimination to get A to be the identity matrix, and whatever emerges to its right is the matrix A^{-1}.

1. (10 points) Consider the initial simplex tableau

BASIC	x_1	x_2	x_3	x_4	x_5	z	*RHS*	θ
x_3	1	2	1	0	0	0	4	$4/2 = 2$
x_4	2	3	0	1	0	0	9	$9/3 = 3$
x_5	8	4	0	0	1	0	16	$16/4 = 4$
	−2	−3	0	0	0	1	0	

Ans. entering variable: x_2 ; departing variable: x_3 ;

BASIC column of new tableau: $[1\ 0\ 0\ 0]^T$;

✓ 10

Bottom line of next tableau: $[-\frac{1}{2}\ \ 0\ \ \frac{3}{2}\ \ 0\ \ 0\ \ 1\ \vdots\ 6]$

(i) (1 point) What is the entering variable? Explain!

x_2, since −3 is the most negative value in objective row $(-3 < -2)$

(ii) (2 points) What is the departing variable? Explain

x_3, b/c the θ-value for x_3 is the smallest compared to x_4 and x_5 $(\frac{4}{2} < \frac{9}{3} < \frac{16}{4})$

(iii) (1 point): What is the BASIC column of the next tableau?

(iv) (6 points): What is the bottom (aka objective) line of the next tableau?

The new x_2 row becomes : $\frac{1}{2}[1\ 2\ 1\ 0\ 0\ 0\ 4]$

$= [\frac{1}{2}\ 1\ \frac{1}{2}\ 0\ 0\ 0\ 2] = r$

To get for last row, multiply r by 3 and add r_f

$\Rightarrow [\frac{3}{2}\ 3\ \frac{3}{2}\ 0\ 0\ 0\ 6]$

$+ [-2\ -3\ 0\ 0\ 0\ 1\ 0]$

$= [-\frac{1}{2}\ 0\ \frac{3}{2}\ 0\ 0\ 1\ 6]$

$3r + r_f \Rightarrow$ bottom line of new tableau

2. (10 points) If the initial simplex tableau was:

BASIC	x_1	x_2	x_3	x_4	x_5	z	RHS
x_3	1	②	1	0	0	0	4
x_4	2	3	0	1	0	0	9
x_5	8	4	0	0	1	0	16
→	−2	−3	0	0	0	1	0

$z - 2x_1 - 3x_2 = 0$

$z = 2x_1 + 3x_2$

$C^T = [2\ 3\ 0\ 0\ 0]$

Check:

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 2 \\ 4 & 1 & 8 \end{bmatrix} \cdot \overset{B^{-1}}{\begin{bmatrix} 2 & -1 & 0 \\ 16 & -12 & 1 \\ -3 & 2 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3\times3}$$

✓

and currently the BASIC column is

$$\begin{bmatrix} x_2 \\ x_5 \\ x_1 \end{bmatrix}$$

What is the x_3 column (inclduing the entry at the bottom line, i.e. what used to be 0) of the current tableau? (Explain!)

✓

Ans.: The $\underline{x_3}$ column of the current tableau is: $\begin{pmatrix} 2 \\ 16 \\ -3 \\ 0 \end{pmatrix}$

$\boxed{10}$

NICR

$$\begin{bmatrix} x_2 \\ x_5 \\ x_1 \end{bmatrix} \Rightarrow \begin{pmatrix} x_2 & x_5 & x_1 \end{pmatrix} \text{ col.} \Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 2 \\ 4 & 1 & 8 \end{bmatrix} = B$$

$$\text{New } x_3 \text{ col} = B^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{pmatrix} 2 & -1 & 0 \\ 16 & -12 & 1 \\ -3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 16 \\ -3 \end{pmatrix}$$

$$B^{-1} \Rightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 6 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 16 & -12 & 1 \\ 0 & 0 & 1 & -3 & 2 & 0 \end{array}\right) \quad \underbrace{}_{} \ \| \ B^{-1}$$

★ Bottom line entry =

$(C^T \cdot \text{new col}) - c_i$

$$= \underset{x_1\ \ x_2\ \ x_3\ \ x_4\ \ x_5}{[2\ \ 3\ \ 0\ \ 0\ \ 0]} \Rightarrow \underset{x_2\ x_5\ x_1}{[3\ 0\ 2]}$$

★

$$[3\ 0\ 2] \begin{bmatrix} 2 \\ 16 \\ -3 \end{bmatrix} = 0 - 0 = 0$$

↑

$c_i x_3 = 0$ in original obj. function.

3. (10 points) A certain linear programming problem with three variables x_1, x_2, x_3, and three constraints, has an optimal solution $x_1 = 0, x_2 = 1, x_3 = 10$, yielding the optimal value 101.

You are also told that its first constraint is **not tight**, i.e. if you plug-in the values of the optimal solution into the first constraint you get a **strict** inequality $<$ (they are **not** equal). Calling the dual variables corresponding to the first, second, and third constraints, w_1, w_2, w_3 respectively, you are also told that the dual constraints are

$$w_1 + w_2 + w_3 \geq 3 \quad , \quad w_1 + 2w_2 + 3w_3 \geq 13 \quad , \quad w_1 + 2w_2 + w_3 \geq 7 \quad .$$

Find (i) (8 points) the optimal solution of the dual problem. (ii) (2 points) the value of the goal function at that optimal solution of the dual problem

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✓ 10

Ans.: The optimal solution of the dual problem is

$w_1 =$ 0 $\quad w_2 =$ 2 $\quad w_3 =$ 3

The value of the goal function at that optimal solution of the dual problem is: 101

Primal:

Dual:

1^{st} constraint SLACK $\neq 0 \Rightarrow w_1 = 0$ $\Rightarrow$

$w_2 + w_3 \geq 3$

$2w_2 + 3w_3 = 13$

$2w_2 + w_3 = 7$

$X_1 = 0 \Rightarrow$ no further conc.

$X_2, X_3 \neq 0 \Rightarrow$ SLACK in dual #2, # 3 constraints $= 0$

$\Downarrow$

$$2w_2 + 3w_3 = 13$$
$$-2w_2 - w_3 = -7$$
$$2w_3 = 6$$
$$w_2 = 2 \quad w_3 = 3$$

4. (10 points)
For the following transportation problem, find the initial basic feasible solution, M, using the **Minimal Cost Rule**. Also find the cost of that solution.

$$\mathbf{C} = \begin{bmatrix} 12 & 5 & 3 \\ 3 & 5 & 7 \\ 7 & 12 & 12 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 30 \\ 31 \\ 11 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 22 \\ 28 \\ 22 \end{bmatrix}.$$

Ans.

$$\mathbf{M} = \begin{bmatrix} 0 & 8 & 22 \\ 22 & 9 & 0 \\ 0 & 11 & 0 \end{bmatrix}$$

✓ (10)

cost= 349

work:

		Start	
$\boxed{12}$	$\boxed{5}$ 8	$\boxed{3}$ 22	30
$\boxed{3}$ 22	$\boxed{5}$ 9	$\boxed{7}$	31
$\boxed{7}$	$\boxed{12}$ 11	$\boxed{12}$	11
22	28	22 =72	=72 ''

$$\begin{array}{r} {}^{1} \\ 40 \\ 66 \\ 66 \\ {}^{2}\,45 \\ +132 \\ \hline 349 \end{array}$$

← $\text{cost} = (8\cdot 5)+(22\cdot 3)+(22\cdot 3)$
$+(5\cdot 9)+(11\cdot 12) = 349$

5. (10 points)
For the following transportation problem, find the initial basic feasible solution, M, using **Vogel's Rule**. Also find the cost of that solution.

$$\mathbf{C} = \begin{bmatrix} 12 & 5 & 3 \\ 3 & 5 & 7 \\ 7 & 12 & 12 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 30 \\ 31 \\ 11 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 22 \\ 28 \\ 22 \end{bmatrix}.$$

Ans.

$$\mathbf{M} = \begin{bmatrix} 0 & 8 & 22 \\ 11 & 20 & 0 \\ 11 & 0 & 0 \end{bmatrix}$$

✓

⑩

cost= 316

12	5	3	2225
3	5	7	2225
7	12	12	⑤
4 ⑨	0 0 0 0	4 4 ④	

	8	22	30
11	20		31
11			11
22	28	22	

Cost = (8·5) + (22·3) + (11·3) + (20·5) + (11·7) =

```
   40
   66
 2 33
  100
+  77
-----
  316
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6. (10 points) Find the permutation matrix, P, that solves the following assignment problem with four employees and four jobs, where C is the cost matrix whose (i, j) entry is the cost of assigning employee i to job j.

$$C = \begin{bmatrix} 6 & 4 & 1 & 5 \\ 2 & 7 & 4 & 8 \\ 2 & 2 & 2 & 2 \\ 13 & 17 & 10 & 18 \end{bmatrix} \begin{matrix} -1 \\ -2 \\ -2 \\ -10 \end{matrix}$$

Ans.:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad 2143$$

✓ 10

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row

col

$$\Rightarrow \begin{bmatrix} 5 & 3 & 0 & 4 \\ 0 & 5 & 2 & 6 \\ 0 & 0 & 0 & 0 \\ 3 & 7 & 0 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 3 & 0^{*} & 4 \\ 0^{*} & 5 & 2 & 6 \\ 0 & 0^{*} & 0 & 0 \\ 3 & 7 & 0 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 3 & 0 & 4 \\ 0 & 5 & 2 & 6 \\ 0 & 0 & 0 & 0 \\ 3 & 7 & 0 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0^{*} & 0 & 1 \\ 0^{*} & 5 & 5 & 6 \\ 0 & 0 & 3 & 0^{*} \\ 0 & 4 & 0^{*} & 5 \end{bmatrix}$$

$a = 3$

$\Rightarrow 2143$

7. (10 points) In a certain transportation problem with four factories and four stores, the current basic feasible solution is

$$M = \begin{bmatrix} 0 & 21 & 0 & 11 \\ 0 & 0 & 34 & 0 \\ 25 & 12 & 13 & \underline{0} \\ 0 & 4 & 0 & 0 \end{bmatrix}.$$

It was found out that the **entering variable** is $\underline{x_{34}}$. find a cheaper solution, M', by performing the relevant step in the Transportation Problem algorithm.

Ans:.

$$M' = \begin{bmatrix} 0 & 32 & 0 & 0 \\ 0 & 0 & 34 & 0 \\ 25 & 1 & 13 & 11 \\ 0 & 4 & 0 & 0 \end{bmatrix}.$$

✓ ⑩

H → V → H → V → ...

↑(0)

$$M = \begin{bmatrix} 0 & \overset{*}{21} & 0 & \overset{-}{11} \\ 0 & 0 & 34 & 0 \\ 25 & \underset{-}{12} & 13 & 0^{*}_{+} \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

Even spots: 12, ⑪

↑ Choose 11 b/c smaller

$$\Rightarrow \begin{bmatrix} 0 & 32 & 0 & 0 \\ 0 & 0 & 34 & 0 \\ 25 & 1 & 13 & 11 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

8. (10 points) In the course of solving an assignment problem with four employees and four jobs, the following partial solution was arrived at:

$$\begin{bmatrix} 0^* & 1 & 2 & 0 \\ 3 & 0 & 4 & 0^* \\ 0 & 5 & 0^* & 4 \\ 2 & 1 & 0 & 3 \end{bmatrix}$$

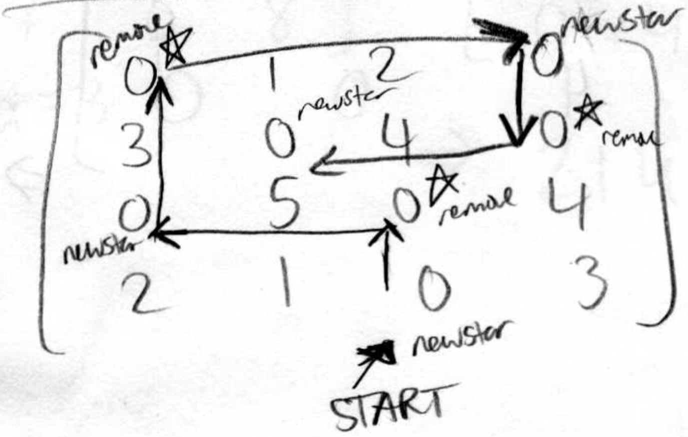
find the permutation matrix that is the final solution.

Ans. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{bmatrix}$ ✓ ⑩

$$\begin{bmatrix} 0 & 1 & 2 & 0^* \\ 3 & 0^* & 4 & 0 \\ 0^* & 5 & 0 & 4 \\ 2 & 1 & \underset{\uparrow}{0^*} & 3 \end{bmatrix} \Rightarrow 4213$$

only option
for row #4

Also could use Alt. pass:



$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 & 0^* \\ 3 & 0^* & 4 & 0 \\ 0^* & 5 & 0 & 4 \\ 2 & 1 & 0^* & 3 \end{bmatrix}$$

9. (20 points altogether) For the following transportation problem.

$$\mathbf{C} = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 3 & 3 \\ 5 & 6 & 6 \end{bmatrix} \quad , \quad \mathbf{s} = \begin{bmatrix} 14 \\ 16 \\ 12 \end{bmatrix} \quad , \quad \mathbf{d} = \begin{bmatrix} 15 \\ 16 \\ 11 \end{bmatrix} \quad .$$

(i) (2 points) Explain why the following solution

$$\begin{array}{c|ccc|c} & W_1 & W_2 & W_3 & \\ \hline V_1 & 0 & 3 & 11 & 14 \\ V_2 & 15 & 1 & 0 & 16 \\ V_3 & 0 & 12 & 0 & 12 \\ \hline & 15 & 16 & 11 & \end{array}$$

is a **basic feasible solution**. Also find its cost.

It's a BFS b/c all the rows + columns add up to corresponding s and d.

$$\Rightarrow s = \begin{bmatrix} 0+3+11 \\ 15+1+0 \\ 0+12+0 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 12 \end{bmatrix} \text{ and } d = \begin{bmatrix} 0+15+0 \\ 3+1+12 \\ 11+0+0 \end{bmatrix} = \begin{bmatrix} 15 \\ 16 \\ 11 \end{bmatrix}$$

(ii) (18 points) Starting with the above basic feasible solution as the initial basic feasible solution, find the optimal solution, and the **minimal cost**.

✓ (20)

Ans.:

Check done:

basic: $\{x_{12}, x_{13}, x_{21}, x_{22}, x_{31}\}$

nonbasic: $\{x_{11}, x_{23}, x_{32}, x_{33}\}$

$$\left.\begin{aligned} v_1+w_2&=5 \\ v_1+w_3&=4 \\ v_2+w_1&=3 \\ v_2+w_2&=3 \\ v_3+w_1&=5 \end{aligned}\right\} \quad \left.\begin{aligned} v_1&=0 \\ v_2&=-2 \\ v_3&=0 \\ w_1&=5 \\ w_2&=5 \\ w_3&=4 \end{aligned}\right\} \quad \begin{aligned} v_1+w_1-6&=-1 \\ v_2+w_3-3&=-1 \\ v_3+w_2-6&=-1 \\ v_3+w_3-6&=-2 \end{aligned}$$

$\underbrace{}_{}$ all ≤ 0 ↗ so done!

$$\begin{bmatrix} 0 & 3 & 11 \\ 3 & 13 & 0 \\ 12 & 0 & 0 \end{bmatrix}$$

minimal cost = 167

$$\begin{array}{r} 15 \\ 44 \\ 9 \\ 39 \\ +60 \\ \hline 167 \end{array}$$

cost = $(3\cdot5)+(11\cdot4)+(3\cdot3)+(13\cdot3)+(12\cdot5)$ //

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①

basic: $\{x_{12}, x_{13}, x_{21}, x_{22}, x_{32}\}$

nonbasic: $\{x_{11}, x_{23}, x_{31}, x_{33}\}$

$$\left.\begin{aligned} v_1+w_2&=5 \\ v_1+w_3&=4 \\ v_2+w_1&=3 \\ v_2+w_2&=3 \\ v_3+w_3&=6 \end{aligned}\right\} \rightarrow \left.\begin{aligned} v_1&=0 \\ v_2&=-2 \\ v_3&=2 \\ w_1&=5 \\ w_2&=5 \\ w_3&=4 \end{aligned}\right\} \begin{aligned} x_{11}&: v_1+w_1-6=-1 \\ x_{23}&: v_2+w_3-3=-1 \\ x_{31}&: v_3+w_1-5=\boxed{2} \\ x_{33}&: v_3+w_3-6=0 \end{aligned}$$

x31 becomes entering var.

even #'s: ⑫, 15

↑ choose 12

$$\Rightarrow \begin{bmatrix} 0 & 3 & 11 \\ 15^{-} & 1^{+} & 0 \\ 0^{+} & 12^{-} & 0 \end{bmatrix}$$

[3,1] → [3,2] → [2,2] → [2,1] → [3,1]

$$\Rightarrow \begin{bmatrix} 0 & 3 & 11 \\ 3 & 13 & 0 \\ 12 & 0 & 0 \end{bmatrix}$$