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MATH 354 (3), Dr. Z. , Exam 2, Mon. Nov. 27, 2023, 10:25-11:35am, TIL-246

FRAME YOUR FINAL ANSWER(S) TO EACH PROBLEM

No Calculators! No books! No Notes! To ensure maximum credit, organize your work neatly and be sure to show all your work.

Do not write below this line

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1. 10 (out of 10)
2. 10 (out of 10)
3. 10 (out of 10)
4. 10 (out of 10)
5. 10 (out of 10)
6. 10 (out of 10)
7. 5 (out of 10)
8. 10 (out of 10)
9. 20 (out of 20)

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tot.: (out of 100)

95

Reminder from Linear Algebra:

The inverse of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\frac{1}{\det(A)}[A_{ij}]^T \quad ,$$

where A_{ij} is $(-1)^{i+j}$ times the determinant of the (i,j) **minor**, which is the $(n-1)\times(n-1)$ matrix obtained by removing the i-th row and the j-th column.

Another way to find the inverse A^{-1} of an $n\times n$ matrix, A, is to stick the identity matrix I_n right after it, getting $[A|I_n]$, perform Gauss-Jordan elimination to get A to be the identity matrix, and whatever emerges to its right is the matrix A^{-1}.

1. (10 points) Consider the initial simplex tableau

BASIC	x_1	x_2	x_3	x_4	x_5	z	RHS	θ-values
x_3	1	2	1	0	0	0	4	$4/2 = 2$
x_4	2	3	0	1	0	0	9	$9/3 = 3$
x_5	8	4	0	0	1	0	16	$16/4 = 4$
	−2	−3	0	0	0	1	0	

departing variable

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Ans. entering variable: ; departing variable: ;

BASIC column of new tableau: $[X_2 \; X_4 \; X_5]^T$;

Bottom line of next tableau: $[-1/2 \quad 0 \quad 3/2 \quad 0 \quad 0 \quad 1 \;|\; 6]$

$-2 + C_B^T \cdot t_1$ $\quad -3 + C_B^T t_2$ $\quad 0 + C_B^T t_3$ $\quad 0 + C_B^T t_4$ $\quad 0 + C_B^T t_5$ $\quad 0 + C_B^T (\text{RHS})$

(i) (1 point) What is the entering variable? Explain!

The entering variable is X_2 since it is the column that has the largest negative value in the last row of the initial simplex tableau.

(ii) (2 points) What is the departing variable? Explain

The departing variable is X_3 since the θ-value corresponding to X_3 (θ for $X_3 = 4/2$) is the smallest positive θ-value.

(iii) (1 point): What is the BASIC column of the next tableau? $[X_2 \; X_4 \; X_5]^T$

(iv) (6 points): What is the bottom (aka objective) line of the next tableau?

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Find B^{-1}:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 \to \frac{r_1}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[r_3 \to r_3 - 4r_1]{r_2 \to r_2 - 3r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -3/2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

B^{-1}

$C_B^T = [3 \quad 0 \quad 0]$

$$t_1 = B^{-1} \cdot X_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ -3/2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 6 \end{bmatrix}$$

$$t_2 = B^{-1} X_2 = \begin{bmatrix} 1/2 & 0 & 0 \\ -3/2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$t_3 = B^{-1} X_3 = \begin{bmatrix} 1/2 & 0 & 0 \\ -3/2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -3/2 \\ -2 \end{bmatrix}$$

$$t_4 = B^{-1} X_4 = \begin{bmatrix} 1/2 & 0 & 0 \\ -3/2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$t_5 = B^{-1} X_5 = \begin{bmatrix} 1/2 & 0 & 0 \\ -3/2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow Z = \begin{bmatrix} 1/2 & 0 & 0 \\ -3/2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$$

2. (10 points) If the initial simplex tableau was:

BASIC	x_1	x_2	x_3	x_4	x_5	z	*RHS*
x_3	1	2	1	0	0	0	4
x_4	2	3	0	1	0	0	9
x_5	8	4	0	0	1	0	16
	-2	-3	0	0	0	1	0

and currently the BASIC column is

$$\begin{bmatrix} x_2 \\ x_5 \\ x_1 \end{bmatrix}$$

What is the x_3 column (inclduing the entry at the bottom line, i.e. what used to be 0) of the current tableau? (Explain!)

$[2 \;\; 16 \;\; -3 \;;\; 0]^T$

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Ans.: The x_3 column of the current tableau is:

$[2 \;\; 16 \;\; -3 \;;\; 0]^T$

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 2 \\ 4 & 1 & 8 \end{bmatrix}$$

To find B^{-1}:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 \to \frac{r_1}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$$

$r_2 \to r_2 - 3r_1$, $r_4 \to r_4 - 4r_1$ $\qquad -\frac{3}{2} + 2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right]$$

$r_2 \to 2r_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \\ 0 & 0 & 1 & -3 & 2 & 0 \end{array}\right]$$

$r_1 \to r_1 - \frac{1}{2}r_3$, $r_2 \to r_2 - 6r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 16 & -12 & 1 \\ 0 & 0 & 1 & -3 & 2 & 0 \end{array}\right]$$

$\underbrace{\hphantom{2 \; -1 \; 0}}_{B^{-1}}$

third column

$$t_3 = B^{-1}(\text{old } x_3) = \begin{bmatrix} 2 & -1 & 0 \\ 16 & -12 & 1 \\ -3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 16 \\ -3 \end{bmatrix}$$

$c_B^T = [3 \;\; 0 \;\; 2]$

bottom entry

$$= 0 + [3 \;\; 0 \;\; 2] \begin{bmatrix} 2 \\ 16 \\ -3 \end{bmatrix} = 6 + 0 - 6 = 0.$$

3. (10 points) A certain linear programming problem with three variables x_1, x_2, x_3, and three constraints, has an optimal solution $x_1 = 0, x_2 = 1, x_3 = 10$, yielding the optimal value 101.

You are also told that its first constraint is **not tight**, i.e. if you plug-in the values of the optimal solution into the first constraint you get a **strict** inequality $<$ (they are **not** equal). Calling the dual variables corresponding to the first, second, and third constraints, w_1, w_2, w_3 respectively, you are also told that the dual constraints are

$$w_1 + w_2 + w_3 \geq 3 \quad , \quad w_1 + 2w_2 + 3w_3 \geq 13 \quad , \quad w_1 + 2w_2 + w_3 \geq 7 \quad .$$

Find (i) (8 points) the optimal solution of the dual problem. (ii) (2 points) the value of the goal function at that optimal solution of the dual problem

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Ans.: The optimal solution of the dual problem is

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$$w_1 = 0 \qquad w_2 = 2 \qquad w_3 = 3$$

The value of the goal function at that optimal solution of the dual problem is: 101

Given, $x_1 = 0$, $x_2 = 1$, $x_3 = 10$
First constraint is not tight
optimal value $= 101$ } For primal problem

$w_1 = 0$.

Since, x_2 and x_3 are non-zero values, the corresponding Second & third constraints in dual problem are tight.

$\Rightarrow 0 + 2w_2 + 3w_3 = 13$ - ①
$\Rightarrow 0 + 2w_2 + w_3 = 7$ - ②

→ Solve: ① - ② ⇒
$2w_3 = 6$
$w_3 = 3$

put this in ①
$\Rightarrow 2w_2 + 9 = 13$
$\Rightarrow 2w_2 = 4$
$\Rightarrow \boxed{w_2 = 2}$

So, the optimal solution of dual problem

$= \boxed{[0 \;\; 2 \;\; 3]}$

Optimal value of dual problem
$=$ optimal value of primal problem
$= \boxed{101}$

4. (10 points)
For the following transportation problem, find the initial basic feasible solution, M, using the **Minimal Cost Rule**. Also find the cost of that solution.

$$\mathbf{C} = \begin{bmatrix} 12 & 5 & 3 \\ 3 & 5 & 7 \\ 7 & 12 & 12 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 30 \\ 31 \\ 11 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 22 \\ 28 \\ 22 \end{bmatrix}.$$

Ans.

$$\mathbf{M} = \begin{bmatrix} 0 & 8 & 12 \\ 22 & 9 & 0 \\ 0 & 11 & 0 \end{bmatrix}$$

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cost= 349

$\boxed{12}$	$\boxed{5}$ 8	$\boxed{3}$ 22	30 (0)
$\boxed{3}$ 22	$\boxed{5}$ 9	$\boxed{7}$	31 (0)
$\boxed{7}$	$\boxed{12}$ 11	$\boxed{12}$	11 (0)
22 (0)	28 (0)	22 (0)	

cost = 5(8) + 3(22) + 3(22) + 5(9) + 12(11)
= 40 + 66 + 66 + 45 + 132
= \$349

5. (10 points)

For the following transportation problem, find the initial basic feasible solution, M, using **Vogel's Rule**. Also find the cost of that solution.

$$\mathbf{C} = \begin{bmatrix} 12 & 5 & 3 \\ 3 & 5 & 7 \\ 7 & 12 & 12 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 30 \\ 31 \\ 11 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 22 \\ 28 \\ 22 \end{bmatrix}.$$

Ans.

$$\mathbf{M} = \begin{bmatrix} 0 & 8 & 22 \\ 11 & 20 & 0 \\ 11 & 0 & 0 \end{bmatrix}$$

cost= 316

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0	8	22	0
11	20	0	0
11	0	0	0
0	0	0	

C =

12	5	3	2
3	5	7	2
~~7~~	~~12~~	~~12~~	5
4	0	4	

$\rightarrow$

12	5	3	2
3	5	7	2
~~7~~	~~12~~	~~12~~	
9	0	4	

$\rightarrow$

~~12~~	~~5~~	~~3~~	2
~~3~~	5	~~7~~	2
~~7~~	12	12	
	0	4	

Cost $= 8(5) + 22(3)$
$+ 11(3) + 20(5)$
$+ 7(11)$
$= 40 + 66 + 33 + 100 + 77$
$= 140 + 99 + 77$
$= \boxed{\$316}$

$$\begin{array}{r} 140 \\ 99 \\ 77 \\ \hline 316 \end{array}$$

6. (10 points) Find the permutation matrix, P, that solves the following assignment problem with four employees and four jobs, where C is the cost matrix whose (i, j) entry is the cost of assigning employee i to job j.

$$C = \begin{bmatrix} 6 & 4 & 1 & 5 \\ 2 & 7 & 4 & 8 \\ 2 & 2 & 2 & 2 \\ 13 & 17 & 10 & 18 \end{bmatrix}$$

10

Ans.:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

6	4	1	5
2	7	4	8
2	2	2	2
13	17	10	18

$\rightarrow$

5	3	0*	4
0*	5	2	6
0	0*	0	0
3	7	0	8

$\rightarrow$

2	0*	0	1
0*	5	5	6
0	0	3	0*
0	4	0*	5

$\downarrow$

Two line notation: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$

One line notation: [2 1 4 3]

7. (10 points) In a certain transportation problem with four factories and four stores, the current basic feasible solution is

$$M = \begin{bmatrix} 0 & 21 & 0 & 11 \\ 0 & 0 & 34 & 0 \\ 25 & 12 & 13 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}.$$

It was found out that the **entering variable** is x_{34}. find a cheaper solution, M', by performing the relevant step in the Transportation Problem algorithm.

Ans:.

$$M' = \begin{bmatrix} 0 & 25 & 0 & 7 \\ 0 & 0 & 34 & 0 \\ 25 & 12 & 13 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

5

+4

0	21	0	11
0	0	34	0
25	12	13	0
0	4	0	0

-4 +4 -4

$[3,4] \to \underbrace{[3,2]}_{4} \to [1,2] \to \underbrace{[1,4]}_{11} \to [3,4]$

4 → smaller value

add and subtract 4 in the cycle

Wrong cycle

⇓

0	25	0	7
0	0	34	0
25	12	13	0
0	0	0	4.

Correct answer (by Dr. Z.)

0	21	0	11
0	0	34	0
25	12	13	0
0	4	0	0

→

0	32	0	0
0	0	34	0
25	1	13	11
0	4	0	0

8. (10 points) In the course of solving an assignment problem with four employees and four jobs, the following partial solution was arrived at:

$$\begin{bmatrix} 0^* & 1 & 2 & 0 \\ 3 & 0 & 4 & 0^* \\ 0 & 5 & 0^* & 4 \\ 2 & 1 & 0 & 3 \end{bmatrix}$$

find the permutation matrix that is the final solution.

Ans.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

✓ 10

0*	1	2	0
3	0	4	0*
0	5	0*	4
2	1	0	0

⇒ finding alternate path

↓

0	1	2	0*
3	0*	4	0
0*	5	0	4
2	1	0*	0

Two line notation:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{bmatrix}$$

One line notation:

$[4\ 2\ 1\ 3]$

9. (20 points altogether) For the following transportation problem.

$$\mathbf{C} = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 3 & 3 \\ 5 & 6 & 6 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 14 \\ 16 \\ 12 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 15 \\ 16 \\ 11 \end{bmatrix}.$$

(i) (2 points) Explain why the following solution

$$\begin{array}{c} \begin{bmatrix} 0 & 3 & 11 \\ 15 & 1 & 0 \\ 0 & 12 & 0 \end{bmatrix} \begin{array}{l} 14 \to 0+3+11=14 \\ 16 \to 15+1=16 \\ 12 \to 0+12+0=12 \end{array} \\ \underline{15 \quad 16 \quad 11} \to 0+15+0=15,\ 3+1+12=16,\ 11+0+0=11 \end{array}$$

is a **basic feasible solution**. Also find its cost.

The given solution is a basic feasible solution because the values in each row add up to each value in the supply vector and the values in each column add up to the corresponding values in the demand vector.

cost $= 3(5)+11(4)+15(3)+3+6(12)$
$=15+44+48+72$
$=59+120$
$=\$179$

(ii) (18 points) Starting with the above basic feasible solution as the initial basic feasible solution, find the optimal solution, and the **minimal cost**.

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Ans.:

$$\begin{bmatrix} 0 & 3 & 11 \\ 3 & 13 & 0 \\ 12 & 0 & 0 \end{bmatrix}$$

minimal cost $= \$167$

$\boxed{6}$ 0	$\boxed{5}$ 3	$\boxed{4}$ 11
$\boxed{3}$ 15	$\boxed{3}$ 1	$\boxed{3}$ 0
$\boxed{5}$ 0	$\boxed{6}$ 12	$\boxed{6}$ 0

Basic variables: $X_{12}, X_{13}, X_{21}, X_{22}, X_{32}$

Nonbasic variables: $X_{11}, X_{23}, X_{31}, X_{33}$

For basic:

$V_1 + W_2 = 5$
$V_1 + W_3 = 4$
$V_2 + W_1 = 3$
$V_2 + W_2 = 3$
$V_3 + W_2 = 6$

$V_1 = 0 \Rightarrow V_1 = 0 \quad W_1 = 5$
$V_2 = -2 \quad W_2 = 5$
$V_3 = 1 \quad W_3 = 4$

For nonbasic variables:

$x_{11}: v_1 + w_1 - c_{11} = 0 + 5 - 6 = -1$

$x_{23}: v_2 + w_3 - c_{23} = -2 + 4 - 3 = -1$

$x_{31}: v_3 + w_1 - c_{31} = 1 + 5 - 5 = 1 \rightarrow$ entering variable is x_{31}

$x_{33}: v_3 + w_3 - c_{33} = 1 + 4 - 6 = -1$

$[3,1] \rightarrow \underbrace{[2,1]}_{15} \rightarrow [2,2] \rightarrow \underbrace{[3,2]}_{12} \rightarrow [3,1]$

$\boxed{6}$ 0	$\boxed{5}$ 3	$\boxed{4}$ 11
$\boxed{3}$ 15	$\boxed{3}$ 1	$\boxed{3}$ 0
$\boxed{5}$ ·0	$\boxed{6}$ 12	$\boxed{6}$ 0

+12 −12 +12 −12

$\Rightarrow$

$\boxed{6}$ 0	$\boxed{5}$ 3	$\boxed{4}$ 11
$\boxed{3}$ 3	$\boxed{3}$ 13	$\boxed{3}$ 0
$\boxed{5}$ 12	$\boxed{6}$ 0	$\boxed{6}$ 0

Basic: $x_{12}, x_{13}, x_{21}, x_{22}, x_{31}$

Nonbasic: $x_{11}, x_{23}, x_{32}, x_{33}$

For basic: $v_1 + w_2 = 5$
$v_1 + w_3 = 4$
$v_2 + w_1 = 3$
$v_2 + w_2 = 3$
$v_3 + w_1 = 5$

$v_1 = 0 \Rightarrow$ $v_1 = 0$ $w_1 = 5$
$v_2 = -2$ $w_2 = 5$
$v_3 = 0$ $w_3 = 4$

For nonbasic:

$x_{11}: v_1 + w_1 - c_{11} = 0 + 5 - 6 = -1$

$x_{23}: v_2 + w_3 - c_{23} = -2 + 4 - 3 = -1$

$x_{32}: v_3 + w_2 - c_{32} = 0 + 5 - 6 = -1$

$x_{33}: v_3 + w_3 - c_{33} = 0 + 4 - 6 = -2$

no more positive values.

Optimal solution:

$\boxed{6}$ 0	$\boxed{5}$ 3	$\boxed{4}$ 11
$\boxed{3}$ 3	$\boxed{3}$ 13	$\boxed{3}$ 0
$\boxed{5}$ 12	$\boxed{6}$ 0	$\boxed{6}$ 0

Optimal cost

$= 5(3) + 4(11) + 3(3) + 3(13) + 5(12$

$= 15 + 44 + 9 + 39 + 60$

$= 15 + 44 + 39 + 69$

$= 59 + 39 + 69$

$= \$167$

$$\begin{array}{r} 59 \\ 39 \\ 69 \\ \hline 167 \end{array}$$