NAME: (print!) ____

E-Mail address: __________________________________

SCC: (please circle): Only I/Only II/I and II/None

MATH 354 (3), Dr. Z. , p Final Exam , Wed., Dec. 20, 2023, 8:00am-11:00am; place: TIL-264 .

WRITE YOUR FINAL ANSWER TO EACH PROBLEM IN THE INDICATED PLACE (right under the question)

Show all your work! No calculators, no cheatsheets

EXPLAIN EVERYTHING

CHECK ALL YOUR ANSWERS! (whenever applicable) . If you did it the right way, but got the wrong answer due to a careless computational error, that you could have detected by checking, you will get NO partial credit.

Do not write below this line

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1. (out of 10)

2. (out of 10)

3. (out of 10)

4. (out of 10)

5. (out of 10)

6. (out of 10)

7. (out of 10)

8. (out of 10)

9. (out of 10)

10. (out of 10)

11. (out of 10)

12. (out of 10)

13. (out of 20)

14. (out of 20)

15. (out of 20)

16. (out of 20)

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tot. (out of 200)

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1. (10 points) Find the rank of the following matrix. Explain! (No credit for just giving the answer.)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix} \quad .$$

Ans.: The rank of the matrix is:

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2. (10 points) Consider the following scenario.

A restaurant chef is planning a meal consisting of three foods: soup, pasta, and pudding.

• The soup has 100 calories per liter, the pasta has 10 calories per kg, and the pudding has 200 calories per kg.

• The soup has 200 grams of protein per liter, the pasta has 50 grams of protein per kg , the pudding has 30 grams of protein per kg.

• The soup has 70 grams of carbohydrates per liter, the pasta has 40 grams of carbohydrates per kg, the pudding has 50 grams of carbohydrates per kg.

The chef wants the meal to consist of

• At least 300 grams of protein.

• At least 400 grams of carbohydrates but at most 600 grams of carbohydrates.

• At most 1000 calories.

The profit that he makes is 2 dollars per liter of soup, 3 dollars per kg of pasta, and 5 dollars per kg of pudding.

Set up, but do **not** solve, a linear programming model of the situation described that would maximize the profits subject to the constraints. Call z the profit and x_1 the amount of soup in liters, x_2 the amount of pasta in kg, and x_3 the amount of pudding in kg.

Ans.:

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3 (10 points) Convert the following linear programming problem into canonical form, by introducing more variables, if necessary. Explain what you are doing. Just giving the right answer, without explanation will earn you only two points).

Minimize $z = 4x_1 + 3x_2 - 4x_3$

Subject to

$$x_1 - x_2 + x_3 \geq 5$$

$$2x_1 - 3x_2 - x_3 \leq 5$$

$x_1 \geq 0$, $x_2 \leq 0$, x_3 unrestricted.

Ans.:

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4. (10 points) Use Vogel's method (no credit for other ways!) to find an initial feasible solution to the following transportation problem. Also find its cost. Explain!

$$\mathbf{C} = \begin{bmatrix} 4 & 6 & 7 & 5 \\ 4 & 4 & 7 & 8 \\ 5 & 3 & 6 & 5 \\ 6 & 5 & 3 & 4 \end{bmatrix} \quad , \quad \mathbf{s} = \begin{bmatrix} 100 \\ 60 \\ 50 \\ 70 \end{bmatrix} \quad , \quad \mathbf{d} = \begin{bmatrix} 40 \\ 70 \\ 120 \\ 50 \end{bmatrix} \quad .$$

Ans.

$$\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \quad ; \qquad Cost = \qquad\qquad .$$

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5. (10 points) Use any method to solve the following linear programming problem.

Maximize $z = 3\,x_1 + 5\,x_2$, subject to the constraints

$$x_1 + 2\,x_2 \leq 6 \quad ,$$

$$2\,x_1 + x_2 \leq 6 \quad ,$$

$$x_1 + x_2 \geq 1 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad .$$

Ans.: $x_1 =$; $x_2 =$; $z =$.

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6. (10 points) Use any method to solve the following linear programming problem.

Minimize $z = 3\,x_1 \,-\, 5\,x_2$, subject to the constraints

$$5\,x_1 \,+\, x_2 \le 50 \quad ,$$

$$x_1 \,+\, 2\,x_2 \le 6 \quad ,$$

$$2\,x_1 \,+\, x_2 \le 6 \quad ,$$

$$x_1 \,+\, x_2 \le 1 \quad ,$$

$$x_1 \ge 0 \quad , \quad x_2 \ge 0 \quad .$$

Ans.: $x_1 =$; $x_2 =$; $z =$.

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7. (10 points) Solve the following linear programming problem

Minimize $z = 3 + x_1 + x_2 + x_3 - x_4 - x_5$ subject to the constraints:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 5 \quad ,$$

$$x_1 + x_2 + x_3 + x_4 = 4 \quad ,$$

$$x_1 + x_2 + x_3 = 3 \quad ,$$

$$x_1 + x_2 = 2 \quad ,$$

$$x_1 + 2x_2 = 3 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \geq 0 \quad , \quad x_4 \geq 0 \quad , \quad x_5 \geq 0 \quad .$$

Ans.:

$x_1 =$; $x_2 =$; $x_3 =$; $x_4 =$; $x_5 =$; $z =$.

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8. (10 points altogether) Consider the following linear programming problem in canonical form.

Maximize $z = x_1 + 2\,x_2 + 3\,x_3 + 4\,x_4$ subject to the constraints:

$$x_1 + x_2 + x_3 + x_4 = 2 \quad ,$$

$$2x_1 + x_2 + x_3 + x_4 = 3 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \geq 0 \quad , \quad x_4 \geq 0 \quad .$$

(i) (4 points) Find the set of **basic solutions**. For each of them indicate the set of basic variables and the set of non-basic variables.

(ii) (2 points) Find the set of **basic feasible solutions**.

(iii) (2 points) Find the optimal solution(s)

(iv) (2 points) Find the optimal value.

Ans. to (i): The set of **basic solutions** (for each, in parentheses, indicate the set of basic variable) is:

Ans. to (ii): The set of **basic feasible solutions** is .

Ans to (iii): An **optimal solution** is $x_1 =$, $x_2 =$, $x_3 =$, $x_4 =$.

Ans. to (iv): The **optimal value** is $z =$.

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9. (10 points) Suppose that $x_1 = 2$, $x_2 = 0$, $x_3 = 4$ is an optimal solution to the following linear programming problem.

Maximize $z = 4x_1 + 2x_2 + 3x_3$,

subject to:

$$2x_1 + 3x_2 + x_3 \leq 12 \quad ,$$

$$x_1 + 4x_2 + 2x_3 \leq 10 \quad ,$$

$$3x_1 + x_2 + x_3 \leq 10 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \geq 0 \quad .$$

Using the principle of complementary slackness (**no credit for other methods!**), and the duality theorem, find an optimal solution to the dual problem (calling the dual variables w_1, w_2, w_3). What value will the objective function of the dual problem have at the optimal solution? **Explain everything!** (only 2 points for the right answers without explanations).

Ans.: $w_1 =$; $w_2 =$; $w_3 =$; optimal value of dual (z') = .

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10. (10 points) In the course of solving an assignment problem with five employees and five jobs, the following partial solution was arrived at:

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 & 0^* \\ 0 & 1 & 0^* & 1 & 1 \\ 0^* & 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 0^* & 9 \end{bmatrix} \quad .$$

By findind an **alternating path** (no credit for inspection or other methods!), find the permutation, P, that is the final solution. Explain! (No credit without explanation).

Ans.:

$$P =$$

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11. (10 points altogether) (a) (8 points) Use the **Minimal Cost Rule** (no credit for other ways!) to find an initial feasible solution to the following transportation problem. (b) (2 points) Find its cost.

$$\mathbf{C} = \begin{bmatrix} 4 & 6 & 7 & 5 \\ 4 & 4 & 7 & 8 \\ 5 & 3 & 6 & 5 \\ 6 & 5 & 3 & 4 \end{bmatrix} \quad , \quad \mathbf{s} = \begin{bmatrix} 100 \\ 60 \\ 50 \\ 70 \end{bmatrix} \quad , \quad \mathbf{d} = \begin{bmatrix} 40 \\ 70 \\ 120 \\ 50 \end{bmatrix} \quad .$$

Ans.

$$\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \quad ; \qquad Cost = \qquad\qquad .$$

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12. (10 points altogether)
(a) (8 points) Solve the assignment problem with the following cost matrix. (b) (2 points) Find the minimal cost.

$$\begin{bmatrix} 3 & 5 & 4 & 2 & 8 & 1 \\ 8 & 3 & 6 & 6 & 4 & 3 \\ 4 & 4 & 8 & 8 & 3 & 5 \\ 3 & 8 & 7 & 4 & 9 & 7 \\ 7 & 7 & 9 & 2 & 3 & 5 \\ 9 & 7 & 2 & 7 & 5 & 8 \end{bmatrix}$$

Ans.:

$$P = \begin{bmatrix} \quad & & & & & \quad \\ & & & & & \\ & & & & & \end{bmatrix} \quad ; \qquad Minimal \quad Cost = \qquad .$$

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13. (20 points altogether) The capacity matrix for the following network with 7 vertices, (where vertex 1 is the source and vertex 7 is the sink (aka terminal)).

$$\mathbf{C} = \begin{bmatrix} 0 & 8 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad .$$

(a) (2 points) Draw a diagram of the network with vertex 1 on the left, followed by vertices 2 and 3 (with vertex 2 located above vertex 3), followed by vertex 4 at the middle, followed by vertices 5 and 6 (with 5 located above 6), followed by vertex 7 on the extreme right.

(b) (10 points) Find the maximal flow on the diagram. You don't have to redraw the diagram at every iteration, but indicate each augmenting path that you are using.

(c) (2 points) Find the value of the maximal flow that you found in part (b).

(d) (5 points) Find a minimum cut

(e) (1 point) Find the value of the minimum cut.

Ans.: (c) Value of maximal flow= ;

(d) Minimal cut: The following set of edges: { } ;

(e) Value of minimal cut = .

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14. (20 points altogether) In the course of using the "keep finding an augmented path until none are left" algorithm to solve the maximum flow problem for a certain network, the following augmented path was found, from the source, vertex 1, to the sink, vertex 9, where the current flow along an edge is indicated in *italics* and its capacity is indicated in **boldface**

$$1 \overset{\mathit{3},\mathbf{7}}{\longrightarrow} 3 \overset{\mathit{5},\mathbf{11}}{\longrightarrow} 2 \overset{\mathit{5},\mathbf{5}}{\longleftarrow} 7 \overset{\mathit{3},\mathbf{5}}{\longleftarrow} 5 \overset{\mathit{6},\mathbf{17}}{\longrightarrow} 9 \quad .$$

(a) (2 points) Explain why this is a legal augmenting path.
(b) (16 points) Find the new flow along each edge.
(c) (2 points) By how much did it increase?

Ans.:

$$1 \overset{\quad,\mathbf{7}}{\longrightarrow} 3 \overset{\quad,\mathbf{11}}{\longrightarrow} 2 \overset{\quad,\mathbf{5}}{\longleftarrow} 7 \overset{\quad,\mathbf{5}}{\longleftarrow} 5 \overset{\quad,\mathbf{17}}{\longrightarrow} 9 \quad .$$

The flow increased by: .

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15. (20 points altogether) For the following linear programming problem

Maximize $z = 2\,x_1 \,+\, x_2$ subject to the restrictions

$$x_1 + 2x_2 \leq 10$$

$$2x_1 + x_2 \leq 5$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0$$

(a) (10 points) Set up an **initial simplex tableau** . Call the slack variables x_3 and x_4.

(b) (2 points) Find the **entering variable** (Explain!).

(c) (2 points) Find the **departing variable** (Explain!).

(d) (6 points) Perform **one** iteration in the simplex algorithm to get the next tableau.

Ans. to (b) and (c) : Entering Variable= ; Departing Variable= .

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16. (20 points altogether) Consider the following linear programming problem in canonical form, with variables x_1, x_2, x_3, x_4.

Maximize $z = x_1 + 3x_2$ subject to

$$x_1 + x_2 + x_3 = 4 \quad ,$$

$$x_1 + x_2 \quad - x_4 = 6 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \geq 0 \quad , \quad x_4 \geq 0 \quad .$$

(a) (4 points) Explain why the usual simplex method would not work.

(b) (2 point) How many artificial variables are needed for the big M method?

(c) (6 points) Calling the artificial variable(s) y_1 (or y_1, y_2 etc., if needed), set up the **initial tableau** for using the big M method.

(d) (4 points) Determine the entering and departing variable of the initial tableau.

(e) (3 points) Perform **one** iteration to get the next tableau.

(f) (1 point) Who is the entering variable right now?