Solutions to Attendance Quiz for Lecture 21

Version of May 12, 2019 (thanks to Smitkumar Patel)

1. In the following network vertex 1 is the source and vertex 5 is the sink (terminal). The capacities are given by the above matrix ($c_{ij} = 0$ means that there is no edge between vertex i and vertex j)

$$\begin{bmatrix} 0 & 4 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Draw the network with the source 1 on the left, the sink (terminal) 5 on the right and vertices 2, 3, and 4 in the middle. Indicate the capacities next to each edge.

For a picture see

`http://sites.math.rutgers.edu/~zeilberg/LinOpt19/p21Spic.pdf`

(b): Verify that the following matrix is a legal flow. What is its strength?

$$\begin{bmatrix} 0 & 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Except for the **source** and the **sink**, that are vertices 1 and 5 respectively, the **flow-in** to each vertex must **equal** to the **flow-out**.

But the flow-out is simply the sum of the corresponding row, and the flow-in is the sum of the corresponding column.

• The sum of row 2 is $0 + 0 + 1 + 0 + 2 = 3$; The sum of column 2 is $3 + 0 + 0 + 0 + 0 = 3$; They are equal, Yea!

• The sum of row 3 is $0 + 0 + 0 + 0 + 3 = 3$; The sum of column 3 is $1 + 1 + 0 + 1 + 0 = 3$; They are equal, Yea!

• The sum of row 4 is $0 + 0 + 1 + 0 + 1 = 2$; The sum of column 4 is $2 + 0 + 0 + 0 + 0 = 2$; They are equal, Yea!

The **strength** of the flow is the total flow **out** of the source, that **must** be (a good check) the **same** as the flow **into** the sink.

Hence the strength of the flow is simply the sum of the first row, that **must** be the same (or you messed up above and it is not a flow) as the sum of the last column.

The sum of the first row is $0+3+1+2+0 = 6$; The sum of the last column is $0+2+3+1+0 = 6$, yea.

Ans. to 2: It is indeed a flow, since indeed the flow out of every vertex (except the source and the sink) happens to be equal to the flow in, and the strength of the flow is 6.

(c) Give two examples of cuts. What are their values?

There are **always** two obvious cuts.

- The set of all the edges that come out of the source, in this example $\{12, 13, 14\}$

- The set of all the edges that go into the sink, in this example $\{25, 35, 45\}$