Solutions to Attendance Quiz for Lecture 19

1. Completely solve the following assignment problem

$$\begin{bmatrix} 5 & 3 & 6 \\ 5 & 5 & 5 \\ 7 & 4 & 8 \end{bmatrix} \quad .$$

Sol. to 1: We find an **equivalent** problem (i.e. a problem with the **same solution**) where every row has at least one zero, by subtracting from each row its smallest entry.

Doing

$$r_1 - 3 \to r_1 \quad , \quad r_2 - 5 \to r_2 \quad , \quad r_3 - 4 \to r_3 \quad ,$$

gives the new problem

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 4 \end{bmatrix} \quad .$$

Next we have to make sure that every column has at least one zero, by subtracting from each column that has no zeros, its smallest entry, like we did with the rows. In this particular problem, each column already has at least one zero, so this step is not necessary (from the point of view of the computer, you can still perform this step, the smallest entry is always 0, and subtracting 0 will not change anything, but we humans can skip this step.)

The next step is to do try and do **match-making**. Alas, this is not possible. Both Column 1 and Column 3 are only willing to marry Row 2, and since bigamy is forbidden, a complete matching is out of the question.

Whenever it is not possible to find a complete matching, it means that all the zeros can be squeezed into less than n lines (in our case $n = 3$). Indeed Row 2 and Column 2 contain all the zeros. These are the **special** lines, that in the book are crossed out. Since I don't like to cross out, I will denote by a_r if an entry a belongs to a special row, a_c if it belongs to a special column, and by a_{rc} if it belongs to both a special row and special column (this corresponds to an intersection of lines in the book).

In this problem we have

$$\begin{bmatrix} 2 & 0_c & 3 \\ 0_r & 0_{rc} & 0_r \\ 3 & 0_c & 4 \end{bmatrix} \quad .$$

The **smallest** entry that is **not** marked is the $(1,1)$ entry, that happens to be 2. We

• **Subtract** 2 from each unmarked entry

so entry $(1,1)$ becomes 0, entry $(1,3)$ becomes 1, entry $(3,1)$ becomes 1, entry $(3,3)$ becomes 2

• **Leave alone** all entries that are only marked by $_r$ or by $_c$

So entries $(1,2),(2,1),(2,3),(3,2)$ stay the same

• **Add** 2 to all entries that are marked with $_{rc}$

So entry $(2,2)$ becomes $0+2=2$.

The new problem is

$$\begin{bmatrix} 0 & 0_c & 1 \\ 0_r & 2_{rc} & 0_r \\ 1 & 0_c & 2 \end{bmatrix} \quad .$$

Removing the distracting $_r$, $_c$, $_{rc}$, we get that the new equivalent problem is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad .$$

Now it is easy to find a perfect matching **by inspection** (but you are welcome to use the official *alternating paths* algorithm), and we get

$$\begin{bmatrix} 0^* & 0 & 1 \\ 0 & 2 & 0^* \\ 1 & 0^* & 2 \end{bmatrix} \quad .$$

Converting the 0^* to 1 and all the other entries to 0, we get the **permutation matrix**, that is a solution to our assignment problem.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad .$$

This correspond to the permutation, in **two-line notation**

$$\begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix}$$

and in **one-line notation** 132

Ans. to 1: The solution to the assignment problem (in one-line notation) is 132