Solutions to REAL Quiz #8 (Dr. Z., Math 250)

1. (4 points) Determine the dimension of the following subspace. Explain what you are doing!

$$\left\{ \begin{bmatrix} 2s \\ -s+4t \\ s-3t \\ s+t \\ s-t \end{bmatrix} \in R^5 \, : \, s \quad and \quad t \quad are \quad scalars \right\} \quad .$$

Sol. of 1: By separating the s part and the t part, We can write a typical vector in this subspace as

$$\begin{bmatrix} 2s \\ -s+4t \\ s-3t \\ s+t \\ s-t \end{bmatrix} = s \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 4 \\ -3 \\ 1 \\ -1 \end{bmatrix} \quad .$$

So our subspace can be written as:

$$\left\{ s \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 4 \\ -3 \\ 1 \\ -1 \end{bmatrix} \, : \, s \quad and \quad t \quad are \quad scalars \right\} \quad ,$$

which is the set of all **linear combinations** of the two vectors $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \\ -3 \\ 1 \\ -1 \end{bmatrix}$, in other words our subspace is their **span**:'

$$V = Span \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad \begin{bmatrix} 0 \\ 4 \\ -3 \\ 1 \\ -1 \end{bmatrix} \right\} \quad .$$

So

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad \begin{bmatrix} 0 \\ 4 \\ -3 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a **generating set**. To find the dimension of this subspace we need to find a **basis**. In general this would mean having to form the matrix whose columns are the vectors, and use Gaussian elimination to find the rank of that matrix. But when you have only **two** vectors, you can easily see that none is a multiple of the other, so they are **linearly independent**, and hence the above generating set is also a basis. Since it has two elements, it means that the dimension is 2.

Ans. to 1: The dimension is 2.

2. (4 points) Below a matrix A and a scalar λ are given. Show that λ is an eignenvalue of the matrix and determine a basis for its eigenspace.

$$A = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \quad , \quad \lambda = 6 \quad .$$

Sol. of 2: We have to show that $A\mathbf{v} = 6\mathbf{v}$ has a non-zero solution. This is the same as solving $(A - (6)I_2)\mathbf{v} = \mathbf{0}$, Now

$$\begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} - 6I_2 = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3-6 & 3 \\ 1 & 5-6 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix} \quad .$$

The fact that

$$\begin{bmatrix} -3 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

In everyday notation means

$$-3v_1 + 3v_2 = 0$$

$$v_1 - v_2 = 0$$

But these two equations are equivalent so we have $v_1 = v_2$, so taking v_2 as the *free— variable, we get* **general solution** *is* $v_1 = v_2 \quad , \quad v_2 = v_2$.

In vector notation this is:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad .$$

Since we got a non-zero solution this means that $\lambda = 6$ *is indeed an eigenvalue and a basis for the* **eigenspace** *is:*

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad .$$