Solutions to REAL Quiz #5 (Dr. Z., Math 250)

1. a) (5 points) Find an LU decomposition of the following matrix

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 3 & 7 \\ 6 & 7 & 17 \end{bmatrix}$$

Sol. of 1a): You use the first phase of Gaussian elimination, **but** only using the elementary row operations of the type $cr_j + r_i \to r_i$, whose **code-name** is $E(i, j; c)$ until you get an **upper triangular** matrix U, taking careful record of these elementary row operations.

Here goes:

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 3 & 7 \\ 6 & 7 & 17 \end{bmatrix} \underset{-3\,r_1 + r_3 \to r_3}{\overset{-2\,r_1 + r_2 \to r_2}{\longrightarrow}} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 4 & 11 \end{bmatrix} \underset{\to}{-4r_2 + r_3 \longrightarrow r_3} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

This is U. To get L we form a 3×3 matrix with 1's on the diagonal, 0 above the diagonal, and

- $L_{2,1} = 2$ (because we used the elementary-row-operation $E(2, 1; -2)$ (i.e. $-2\,r_1 + r_2 \to r_2$)
- $L_{3,1} = 3$ (because we used the elementary-row-operation $E(3, 1; -3)$ (i.e. $-3\,r_1 + r_3 \to r_3$)
- $L_{3,2} = 4$ (because we used the elementary-row-operation $E(3, 2; -4)$ (i.e. $-4\,r_2 + r_3 \to r_3$)

So we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \quad .$$

Ans. to 1a):

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \quad . \quad , \quad U = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \quad .$$

b) (3 points) Use the answer to part a) to solve the following system of linear equations:

$$2x_1 + x_2 + 2x_3 = 5$$

$$4x_1 + 3x_2 + 7x_3 = 14$$

$$6x_1 + 7x_2 + 17x_3 = 30$$

Sol. of 1b): We have to solve $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 3 & 7 \\ 6 & 7 & 17 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 5 \\ 14 \\ 30 \end{bmatrix} \quad .$$

We now know that $A = LU$ so we have to solve $LU\mathbf{x} = \mathbf{b}$. Putting $\mathbf{y} = U\mathbf{x}$, we have $L\mathbf{y} = \mathbf{b}$. So we first have to solve

$$L\mathbf{y} = \mathbf{b} \quad .$$

In everyday notation this is

$$y_1 = 5 \quad ,$$

$$2\,y_1 + y_2 = 14 \quad ,$$

$$3\,y_1 + 4\,y_2 + y_3 = 30 \quad .$$

We get $y_1 = 5$ for free. Plugging-into the second equation we get $2(5) + y_2 = 14$ so $y_2 = 14 - 2 \cdot 5 = 14 - 10 = 4$ and finally $3\,(5) + 4(4) + y_3 = 30$ gives $y_3 = 30 - 15 - 16 = -1$. So

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix} \quad .$$

Now we are ready to solve $U\mathbf{x} = \mathbf{y}$. (Using the y_1, y_2, y_3 that we have just found). In everyday notation this is:

$$2\,x_1 + x_2 + 2\,x_3 = 5$$

$$x_2 + 3x_3 = 4$$

$$-x_3 = -1 \quad .$$

Going **bottom up**, we have $x_3 = 1$, $x_2 = 4 - 3\,x_3 = 4 - 3 \cdot 1 = 4 - 3 = 1$, $2\,x_1 = 5 - x_2 - 2x_3 = 5 - 1 - 2 \cdot 1 = 2$ and so $x_1 = 1$.

Ans. to 1b):

$$x_1 = 1 \quad , \quad x_2 = 1 \quad , \quad x_3 = 1 \quad .$$