Solutions to REAL Quiz #10

1. (8 points) Find the QR decomposition of the matrix

$$A = \begin{bmatrix} \sqrt{2} & \frac{7\sqrt{2}}{2} \\ \sqrt{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \quad .$$

Sol. of 1 First we must find an orthonormal basis for the column space of A. The input is

$$\mathbf{a_1} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \quad , \quad \mathbf{a_2} = \begin{bmatrix} \frac{7\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \quad .$$

Now

$$\mathbf{v_1} = \mathbf{a_1} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \quad , \qquad ,$$

$$\mathbf{v_2} = \mathbf{a_2} - (\frac{\mathbf{a_2}.\mathbf{v_1}}{||\mathbf{v_1}||^2})\mathbf{v_1}$$

We have

$$\mathbf{a_2}.\mathbf{v_1}$$

$$= \begin{bmatrix} \frac{7\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} . \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$= (\frac{7\sqrt{2}}{2})(\sqrt{2}) + (-\frac{\sqrt{2}}{2})(\sqrt{2}) = 7 - 1 = 6 \quad .$$

$$||\mathbf{v_1}||^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 4 \quad .$$

Hence

$$\mathbf{v_2} = \mathbf{a_2} - \frac{6}{4}\mathbf{v_1} = \begin{bmatrix} \frac{7\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} - \frac{3}{2}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{7\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} - \begin{bmatrix} \frac{3\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

Hence, an **orthogonal** set with the same span is

$$\mathbf{v_1} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \quad , \quad \mathbf{v_2} = \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix} \quad .$$

Next we have to find an **orthonormal** basis, by **normailizing** $\mathbf{v_1}$ and $\mathbf{v_2}$.

$$||\mathbf{v_1}|| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2 \quad .$$

$$||\mathbf{v_2}|| = \sqrt{(2\sqrt{2})^2 + (-2\sqrt{2})^2} = \sqrt{16} = 4 \quad .$$

Hence

$$\mathbf{w_1} = \frac{1}{||\mathbf{v_1}||}\mathbf{v_1} = \frac{1}{2}\begin{bmatrix} \sqrt{2} \\ \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\mathbf{w_2} = \frac{1}{||\mathbf{v_2}||}\mathbf{v_2} = \frac{1}{4}\begin{bmatrix} 2\sqrt{2} \\ \\ -2\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

So the output for Gram-Schmidt, after the **normalizations** is:

$$\mathbf{w_1} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \\ \frac{\sqrt{2}}{2} \end{bmatrix} \quad , \quad \mathbf{w_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

To get Q we just form the matrix whose two columns are $\mathbf{w_1}$ and $\mathbf{w_2}$:

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \quad .$$

Now R is the upper triangular matrix

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} \quad ,$$

where

$$r_{11} = \mathbf{w_1}.\mathbf{a_1} \quad , \quad r_{12} = \mathbf{w_1}.\mathbf{a_2} \quad , \quad r_{22} = \mathbf{w_2}.\mathbf{a_2} \quad .$$

We have

$$r_{11} = (\frac{\sqrt{2}}{2})(\sqrt{2}) + (\frac{\sqrt{2}}{2})(\sqrt{2}) = 1 + 1 = 2 \quad .$$

$$r_{12} = (\frac{\sqrt{2}}{2})(\frac{7\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2})(-\frac{\sqrt{2}}{2}) = 7/2 - 1/2 = 3 \quad .$$

$$r_{22} = (\frac{\sqrt{2}}{2})(\frac{7\sqrt{2}}{2}) + (-\frac{\sqrt{2}}{2})(-\frac{\sqrt{2}}{2}) = 7/2 + 1/2 = 4 \quad .$$

Hence

$$R = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \quad .$$

Answer to the problem: The QR decomposition of A is

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \quad , \quad R = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \quad .$$

You should check that indeed $QR = A$.