Solutions to the Attendance Quiz for Lecture 4

1. Determine whether the given system is consistent, and if so, find its general solution.

$$x_1 + 3x_2 + x_3 + x_4 = -1$$

$$-2x_1 - 6x_2 - x_3 = 5$$

$$x_1 + 3x_2 + 2x_3 + 3x_4 = 2 \quad .$$

Sol. of 1: The **augmented matrix** is:

$$\begin{bmatrix} 1 & 3 & 1 & 1 & -1 \\ -2 & -6 & -1 & 0 & 5 \\ 1 & 3 & 2 & 3 & 2 \end{bmatrix} \quad .$$

We first transform it, using **elementary row operations**, to *row-echelon form.* The left-most non-zero entry of the first row is non-zero, so we don't have to do any swapping. Now we perform $r_2 + 2r_1 \to r_2$ and $r_3 - r_1 \to r_3$ to get the entries under the first-row-pivot to be 0:

$$\begin{bmatrix} 1 & 3 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \quad ,$$

Ignoring the first row, we have two columns of $\mathbf{0}$ so the pivot of the second-row is at the third-column (i.e. it is the $(2,3)$-entry). We want to make the entries under it 0. So we perform the elementary row operation $r_3 - r_2 \to r_3$ getting:

$$\begin{bmatrix} 1 & 3 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad .$$

This is **now** in **row-echelon-form**. The next phase is to bring it to **reduced** row-echelon form. Now we go **bottom-up**. The third row is all 0's and has no pivots, so we leave it alone. Right now above the pivot on the second-row (entry $(2,3)$) there is a 1 (in entry $(1,3)$), so we perform the elementary row-operation $r_1 - r_2 \to r_1$, getting:

$$\begin{bmatrix} 1 & 3 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad .$$

Now it is in **reduced-row-echelon form**. Since column 1 and column 3 have *pivots*, x_1 and x_3 are the **basic variables**, and the other variables (x_2 and x_4) are **free variables**. Traslating to everyday language

$$x_1 + 3x_2 - x_4 = -4 \quad ,$$

$$x_3 + 2x_4 = 3 \quad .$$

(We don't write $0 = 0$, this gives us no information we didn't know before). Expressing the basic variables in terms of the free variables, we have

$$x_1 = -4 - 3x_2 + x_4 \quad ,$$

$$x_2 = x_2$$

$$x_3 = 3 - 2x_4 \quad .$$

$$x_4 = x_4$$

This is the **general solution** (and the system is consistent), and you would have gotten full credit for such an answer. Sometimes you are also asked to write the general solution in **vector form**. Then you would continue:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 - 3x_2 + x_4 \\ x_2 \\ 3 - 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad .$$

Comments: About 50% of the people got it completely right. Most peole were on the right track, but either didn't have time to finish, or messed up the arithmetic.

2. Find the rank and nullity of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 0 & -4 & 1 \end{bmatrix} \quad .$$

Sol. of 2: We transform it to row-echelon-form: Doing $r_2 - r_1 \rightarrow r_2$ and $r_3 - 2r_1 \rightarrow r_3$ yields:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & -2 & -6 & -1 \end{bmatrix} \quad .$$

Doing $2r_2 + r_3 \rightarrow r_3$ yields

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad .$$

Now we look at the **number of non-zero rows**. Since none of the rows have all 0's, that number is 3. This is the **rank** of A. the **nullity** is $n - rank(A) = 4 - 3 = 1$.

Ans.: The rank of A is 3 and the nullity of A is 1.

Comments: **1**. To find out the rank (and hence the nullity) it is enough to go to row-echelon-form (i.e. no need to make it reduced-row-echelon). It is not a mistake to go all the way to the reduced-row-echelon form but it is a waste of time!

2. Don't confuse the two types of problems. Quite a few people said that the system is inconsistent. They would have been right if the matrix of the question would have been the augmented matrix of a system of equations, but this problem, of finding the rank and the nullity has nothing to do with solving systems.

3. About %40 of the poeple got it completely, quite a few almost got it, but then answered "inconsistent" which is not what has been asked. Quite a few people ran out of time, partly because they tried to transform it to reduced-row-echelon form.