Solutions to the Attendance Quiz for Lecture 21

1. (a) Apply the Gram-Schmidt process to replace the given linearly independent set $\mathcal{S}$ by an orthogonal set of non-zero vectors with the same span.

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad , \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Sol. of 1: The data is

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad , \quad \mathbf{u_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Gram-Schmidt says

$$\mathbf{v_1} = \mathbf{u_1} \quad .$$

$$\mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{u_2} \cdot \mathbf{v_1}}{||\mathbf{v_1}||^2} \mathbf{v_1} \quad .$$

So

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad .$$

We have

$$\mathbf{u_2} \cdot \mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = (1)(1) + (-2)(-1) + (1)(0) = 1 + 2 + 0 = 3,$$

and

$$||\mathbf{v_1}||^2 = (1)^2 + (-2)^2 + (1)^2 = 6 \quad .$$

Putting it together, we have:

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

Ans. to 1a: An orthogonal basis for the span of $\mathcal{S}$ is:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad , \quad \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \right\} \quad .$$

(b) Obtain an orthonormal set with the same span as $\mathcal{S}$.

Sol. of 1b): We divide these two vectors by their norms. We have

$$||\mathbf{v_1}|| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6} \quad ,$$

$$||\mathbf{v_2}|| = \sqrt{(\frac{1}{2})^2 + (0)^2 + (\frac{1}{2})^2} = \frac{\sqrt{2}}{2} \quad ,$$

So

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{||\mathbf{v_1}||} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{2\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \quad ,$$

$$\mathbf{w_2} = \frac{\mathbf{v_2}}{||\mathbf{v_2}||} = \sqrt{2} \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \quad .$$

Ans. to 1b: An orthonormal basis for the span of $\mathcal{S}$ is:

$$\left\{ \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{2\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \quad , \quad \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} . \right\} \quad .$$