Solutions to the Attendance Quiz for Lecture 17

1. A matrix and a vector are given. Show that the vector is an eigenvector of the matrix, and determine the corresponding eigenvalue.

$$A = \begin{bmatrix} -9 & -8 & 5 \\ 7 & 6 & -5 \\ -6 & -6 & 4 \end{bmatrix} \quad , \quad \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \quad .$$

Sol. to 1: We multiply the matrix A by the given vector and see whether we get a **multiple** of that vector.

$$A = \begin{bmatrix} -9 & -8 & 5 \\ 7 & 6 & -5 \\ -6 & -6 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} (-9)(3) + (-8)(-2) + (5)(1) \\ (7)(3) + (6)(-2) + (-5)(1) \\ (-6)(3) + (-6)(-2) + (4)(1) \end{bmatrix} = \begin{bmatrix} -27 + 16 + 5 \\ 21 - 12 - 5 \\ -18 + 12 + 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ -2 \end{bmatrix} \quad .$$

Obviously this vector is a multiple (by -2) to the original vector $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} -6 \\ 4 \\ -2 \end{bmatrix} = (-2) \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \quad ,$$

so the proposed vector is indeed an **eigenvector** of the matrix A and the corresponding **eigenvalue** is -2.

2. Below a matrix and a scalar λ are given. Show that λ is an eignenvalue of the matrix and determine a basis for its eigenspace.

$$A = \begin{bmatrix} -11 & 14 \\ -7 & 10 \end{bmatrix} \quad , \quad \lambda = -4 \quad .$$

Sol. of 2: We have to see whether the equation $A\mathbf{x} = (-4)\mathbf{x}$ has a non-zero solution, or in other words, we have to solve the system $(A - (-4)I_2)\mathbf{x} = \mathbf{0}$. If the only solution is the zero vector, then it is **not** an eigenvalue. On the other-hand if we can find a non-zero solution then it is, and the set of solutions will be the **eigenspace**.

$$A - (-4)I_2 = \begin{bmatrix} -11 & 14 \\ -7 & 10 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -11 + 4 & 14 \\ -7 & 10 + 4 \end{bmatrix} = \begin{bmatrix} -7 & 14 \\ -7 & 14 \end{bmatrix}$$

Doing Gaussian elimination we get

$$\begin{bmatrix} -7 & 14 \\ -7 & 14 \end{bmatrix} \overset{r_2 - r_1 \to r_2}{\longrightarrow} \begin{bmatrix} -7 & 14 \\ 0 & 0 \end{bmatrix} \overset{(-1/7)r_1 \to r_1}{\longrightarrow} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad .$$

This is in **reduced-row-echelon form**. In everyday notation this is:

$$x_1 - 2x_2 = 0 \quad .$$

So the general solution is:

$$x_1 = 2x_2$$

$$x_2 = x_2 \quad (free)$$

In vector notation this is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad .$$

So the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an **eigenvector**, but so are all its multiples, and the **eigenspace** is

$$Span \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \quad .$$

Ans. to 2: $\lambda = -4$ is indeed an eigenvalue and a **basis** for its **eigenspace** is

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} .$$

Note: Don't confuse the **eigenspace** that is a **subspace** that has infinitely many inhabitants, with its **basis** that only has finitely many members, in this example, just one, since the dimension of the eignespace is 1.

Comment : For 2×2 matrices it may be easier to do the 'high school' way, like I did in class.