Solutions to the Attendance Quiz for Lecture 11

1. a) Find an LU decomposition of the following matrix

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & -1 & 4 \\ -2 & 1 & 2 \end{bmatrix} \quad .$$

Sol. of 1a): We apply the first phase of **Gaussian Elimination** bringing it to **row-echelon form**, keeping track of the elementary row operations that we performed.

To make the $(2,1)$-entry be 0 and the $(3,1)$ entry be 0, we perform

$$(-2)r_1 + r_2 \to r_2 \quad , \quad r_1 + r_3 \to r_3 \quad .$$

Getting

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad .$$

To our great delight this is already in row-echelon form, and this matrix is our U. To get L we look at the above elementary row operations.

Recall that the 'code-name' of the elementary row operation $cr_j + r_i \to r_i$ is $E(i,j;c)$, and for each such, $L_{i,j} = -c$.

The 'code-name' of the elementary row operation

$$(-2)r_1 + r_2 \to r_2 \quad ,$$

$E(2,1;-2)$. so $c = -2$ and $L_{2,1} = -c = -(-2) = 2$.

The 'code-name' of

$$r_1 + r_3 \to r_3 \quad .$$

is $E(3,1;1)$ so $c = 1$ and $L_{3,1} = -c = -(1) = -1$.

Since $L_{3,2}$ does not show up (there is no elementary row operation of the form $cr_2 + r_3 \to r_3$), we have $L_{3,2} = 0$. Of course, all entries **on the diagonal** are 1 and all the entires **above the diagonal** are 0. Hence:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad .$$

Ans. to 1:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad , \quad U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

b) Use the answer to part a) to solve the following system of linear equations:

$$2x_1 - x_2 + x_3 = -1$$

$$4x_1 - x_2 + 4x_3 = -2$$

$$-2x_1 + x_2 + 2x_3 = -2$$

Sol. of 1b): In matrix notation, we have to solve the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & -1 & 4 \\ -2 & 1 & 2 \end{bmatrix}$$

is the matrix of part a), and the right-hand-side vector $\mathbf{b}$ is given by:

$$\mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} \quad .$$

In part a) we found L and U such that $A = LU$, for a certain **lower-triangular matrix** L, and another **upper-triangular matrix** U.

So the system $A\mathbf{x} = \mathbf{b}$ can be written as:

$$LU\mathbf{x} = \mathbf{b}$$

Now define

$$\mathbf{y} = U\mathbf{x} \quad .$$

Then we have the **two** systems

$$L\mathbf{y} = \mathbf{b} \quad , \quad U\mathbf{x} = \mathbf{y} \quad .$$

This is a **two-step** process.

We first solve $L\mathbf{y} = \mathbf{b}$ getting the vector $\mathbf{y}$. In everyday language this is

$$y_1 = -1$$

$$2y_1 + y_2 = -2$$

$$-y_1 + y_3 = -2 \quad .$$

We use **forward-substitution** going **top-down**. From the first equation we immediately know that $y_1 = -1$. Plugging this information into the second equation we get $2(-1) + y_2 = -2$ that tells us that $y_2 = 0$. Finally we go to the third equation (that would normally involve both y_1 and y_2, it is a fluke that in this problem y_2 is absent), and get $-(-1) + y_3 = -2$ that tells us $1 + y_3 = -2$ so $y_3 = -3$. So the solution of the first system is

$$\mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix} \quad .$$

Now we have to solve the system $U\mathbf{x} = \mathbf{y}$, i.e.

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix} \quad .$$

In everyday notation this means

$$2x_1 - x_2 + x_3 = -1$$

$$x_2 + 2x_3 = 0$$

$$3x_3 = -3 \quad .$$

Now we use **back-substitution** going **bottom-up**.

From the last equation we get

$$x_3 = -1 \quad .$$

Plugging this information into the second equation we get:

$$x_2 + 2 \cdot (-1) = 0 \quad ,$$

so

$$x_2 = 2 \quad .$$

Plugging the already-known values of x_3 and x_2 into the first equation we get

$$2x_1 - 2 + (-1) = -1 \quad ,$$

that yields $x_1 = 1$. Hooray we have solved the system. **Ans. to 1b)**:

$$x_1 = 1 \quad , \quad x_2 = 2 \quad , \quad x_3 = -1 \quad .$$

In vector notation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad .$$

You are welcome to check the answer by plugging-in into the system.