**Math 250**
**Fall 2019**
**Midterm II**
**November 14, 2019**
**Time Limit: 80 Minutes**

**Name (Print):** Answer Key

NetID ______________

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Welcome to your Midterm! You have 80 minutes to take this exam, for a total of 100 points. No books, notes, calculators, cellphones or any kind of electronic device are allowed. `Remember that you are not only graded on your final answer, but also on your work. Thus, unless stated otherwise, you MUST justify your answers.`

This exam contains 14 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

Several pages have boxes. You MUST state your final answer to the problem that is stated in that page inside the box. If the final answer is not in this box, it will be considered as incomplete, even if the final answer is in a different place.

If you need to continue your work on a different page, clearly indicate so, or else your work will be discarded. Do not write in the table that is on the second page of this midterm. There is an extra sheet at the end of this test that you can use if you need more space. If the extra page is not enough extra-space for you, please, let me know so I can give you another paper sheet.

**Academic Honesty Statement:** I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to the academic dishonesty policy and procedures of the Academic Integrity of Rutgers University Statement.

**Signature:** ______________

| Problem | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 15 | |
| 3 | 10 | |
| 4 | 20 | |
| 5 | 11 | |
| 6 | 10 | |
| 7 | 22 | |
| Total: | 100 | |

1. (12 points) Given the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & 1 & -2 \\ -1 & 0 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0 & 1 & 3 & -1 \\ -1 & 2 & 1 & 0 \\ 1 & -3 & -2 & 4 \end{bmatrix}$$

Determine

(a) (3 points) Dimension of $\boxed{Col(A) = \ 3}$

(b) (3 points) Dimension of $\boxed{Null(A) = \ 0}$

(c) (3 points) Dimension of $\boxed{Row(A) = \ 3}$

(d) (3 points) Dimension of $\boxed{Null(A^T) = \ 1}$

Do not forget to show your work!

We must find the rank(A). Since ~~A~~ a 3x4 matrix is easier to handle than a 4x3 matrix, lets compute rank($A^T$).

$$A^T \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -3 & -2 & 4 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{bmatrix} \xrightarrow{R_1 + R_2 \mapsto r_2} \begin{bmatrix} 1 & -3 & -2 & 4 \\ 0 & -1 & -1 & 4 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_3 \mapsto R_3} \begin{bmatrix} 1 & -3 & -2 & 4 \\ 0 & -1 & -1 & 4 \\ 0 & 0 & 2 & 3 \end{bmatrix} = U \quad \text{(row echelon form)}$$

U has 3 pivots $\Rightarrow \boxed{\text{rank}(A^T) = \text{rank}(A) = 3}$

Thus $\boxed{\dim(\text{Row}(A)) = \text{rank}(A^T) = 3 = \text{rank}(A) = \dim(\text{Col}(A))}$

$\text{Nullity}(A) = \dim(\text{Null}(A))$ and $\text{Nullity}(A) = 3 - \text{rank}(A) = 0$

$\Rightarrow \boxed{\dim(\text{Null}(A)) = 0}$

$\text{Nullity}(A^T) = \dim(\text{Null}(A^T))$ and $\text{Nullity}(A^T) = 4 - \text{rank}(A^T) = 1$

$\Rightarrow \boxed{\dim(\text{Null}(A^T)) = 1}$

2. (15 points) (a) ( 3 points) Define a subspace of $\mathbb{R}^n$, by listing the three properties that it must have.

A subset $V$ of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if the following three properties hold.

(i) $\vec{0} \in V$

(ii) $\vec{u}, \vec{v} \in V \Rightarrow \vec{u}+\vec{v} \in V$

(iii) $\vec{u} \in V$ and $k \in \mathbb{R} \Rightarrow k\vec{u} \in V$.

(b) ( 6 points) By checking each of the three properties that a subset of $\mathbb{R}^n$ must have (hopefully you listed them in part (a)), explain why $V$ (given below) is a subspace of $\mathbb{R}^4$.

$$V = \left\{ \begin{bmatrix} r+s \\ r-s \\ 3r+2s \\ r \end{bmatrix} \in \mathbb{R}^4 | r, s \in \mathbb{R} \right\}. = \left\{ r\begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^4 ; r, s \in \mathbb{R} \right\}$$

Span notation $= \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

(i) $\vec{0} \in V$: Let $r=s=0$. Thus, $\begin{bmatrix} r+s \\ r-s \\ 2r+2s \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(ii) $\vec{u}, \vec{v} \in V \Rightarrow$ exist $r_1, r_2, s_1, s_2$ such that

$$\vec{u} = r_1 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \vec{v} = r_2 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{u}+\vec{v} = (r_1+r_2)\begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + (s_1+s_2)\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}.$$

~~Because~~ ~~let~~ Letting $r = r_1+r_2$ and $s = s_1+s_2$ property (ii) holds.

(iii) $\vec{u} \in V$, $c \in \mathbb{R} \Rightarrow$ exist $r_1, s_1 \in \mathbb{R}$ such that

$$\vec{u} = r_1 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow c\vec{u} = c\left( r_1 \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right) = (cr_1)\begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + (cs_1)\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

Letting $r = cr_1$, $s = cs_1$ property (iii) hods.

(c) ( 6 points) We saw during class all the needed steps to show that a set $\mathcal{B}$ is a basis for a subspace $V$ of $\mathbb{R}^n$. Find a basis for $V$. Do NOT forget to show that the basis you found is indeed a basis!!!

Since $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\} = \text{span}\{\vec{V}_1, \vec{V}_2\}$

then $\mathcal{B} = \{\vec{V}_1, \vec{V}_2\}$ generates $V$. ~~Thus~~
Being $\mathcal{B}$ the generating set for $V$, then (obviously) $\mathcal{B}$ is contained in $V$.
It remains to show that the set $\mathcal{B}$ is linearly independent. By inspection, we notice that $\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$ is NOT a multiple of $\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$ (because, in particular, there is not constant $c \in \mathbb{R}$ such that the 4th entry of $\vec{V}_1$ is equals to the fourth entry of $c\vec{V}_2$).

3. (10 points) Find $\det(A)$, where

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 1 \\ 1 & 0 & 0 & 4 \end{bmatrix}.$$

Because the $2^{nd}$ row has 3-zeros I will use cofactor expansion on second row.

Thus,

$$\det(A) = 1 \cdot \det\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & 0 & 4 \end{bmatrix}\right)$$

by Cofactor expansion on row 3

$$= (-1)^4(1)\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + (-1)^6(4)\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 4 + 4(1) = 8$$

Method 2:

$$A \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & 5 \end{bmatrix}$$

① $-2R_1 + R_3 \mapsto R_3$
② $-R_1 + R_4 \mapsto R_4$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 8 \end{bmatrix} = U$$

③ $R_3 + R_4 \mapsto R_4$

$$\Rightarrow \det(A) = (-1)^0 \det(U) = 8$$

Using the row echelon form of A

$\boxed{\det(A) = 8}$

(b) (10 points) For what values of $c$ is the following matrix **NOT** invertible.

$$A = \begin{bmatrix} 2-c & 1 & 1 & 1 \\ 2-c & 2-c & c & c \\ 0 & 0 & 3+c & 0 \\ 0 & 0 & 0 & c-1 \end{bmatrix}$$

Recall that $A$ is invertible if and only if $\det(A) \neq 0$.

$\det(A) = (2-c)(1-c)(3+c)(c-1) = (c-1)^2(c-2)(c+3)$

Thus, $A$ not invertible if and only if $\det(A) = 0$

~~Recall that~~ ~~an nxn matrix is inve~~

Therefore, in order for $A$ to be not invertible, $c$ must take the values $1, 2$ or $-3$

$$A \xrightarrow{-R_1+R_2 \mapsto R_2} \begin{bmatrix} 2-c & 1 & 1 & 1 \\ 0 & 1-c & c-1 & c-1 \\ 0 & 0 & 3+c & 0 \\ 0 & 0 & 0 & c-1 \end{bmatrix}$$

$$\boxed{\text{A not invertible for } c=1,\ c=2 \text{ or } c=-3.}$$

4. (20 points) Find $A^{-1}$ (or say 'A is not invertible') where:

(a) (10 points)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{(2) } -3R_1+R_3 \mapsto R_3]{\text{(1) } -R_1+R_2 \mapsto R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -7 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{(4) } 7R_2+R_3 \mapsto R_3]{\text{(3) } -\frac{1}{2}R_2 \mapsto R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & +\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 & \frac{1}{2} & -\frac{7}{2} & 1 \end{array}\right] \xrightarrow{\substack{\text{(5) } R_3+R_1 \mapsto R_1 \\ \text{(6) } -R_3 \mapsto R_3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{3}{2} & -\frac{7}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{2} & -1 \end{array}\right]$$

$$\xrightarrow[\text{(7) } -2R_2+R_1 \to R_1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{5}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{2} & -1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{7}{2} & -1 \end{bmatrix}}$$

5. (11 points) For the following problems, you must tell **WHY** (you can tell quicky that) the set $\mathcal{S}$ is not a basis of the given $\mathbb{R}^n$ space.

(a) (4 points)

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 9 \\ -11 \\ 13 \\ 22 \\ -1 \end{bmatrix}, \begin{bmatrix} 19 \\ 31 \\ 13 \\ 12 \\ 3 \end{bmatrix} \right\} \subset \mathbb{R}^5$$

Every basis of $\mathbb{R}^5$ has exactly 5 vectors. Since $S$ has 4 vectors, ~~ther~~ it is impossible for $S$ to be a basis of $\mathbb{R}^5$.

(b) (4 points)

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ -11 \\ 4 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} 9 \\ -11 \\ 13 \\ 22 \\ -1 \end{bmatrix}, \begin{bmatrix} 19 \\ 31 \\ 13 \\ 12 \\ 3 \end{bmatrix} \right\} \subset \mathbb{R}^5$$

$S$ has 6 vectors, but every basis of $\mathbb{R}^5$ must have exactly 5 vectors. Thus, it is impossible for $S$ to be a basis of $\mathbb{R}^5$.

(c) (3 points)

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^2$$

A basis is a linear independent set. However,

$$\begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, $S$ is a linear dependent set. Thus, it is impossible for $S$ to be a basis of $\mathbb{R}^2$.

6. (10 points) Does the matrix $A$ have $LU$ decomposition? If it does, find $L$ and $U$. If $A$ does not have an $LU$-decomposition say so but justify your answer (state what is failing).

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow[\text{(2) } -2R_1+R_3 \mapsto R_3]{\text{(1) } -2R_1+R_2 \mapsto R_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & 1 \end{bmatrix} \xrightarrow{\text{(3) } -3R_2+R_3 \mapsto R_3} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(1) $\Rightarrow$ $l_{21} = 2$

(2) $\Rightarrow$ $l_{31} = 2$

(3) $\Rightarrow$ $l_{32} = 3$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_2+R_3 \mapsto R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{2R_1+R_3 \mapsto R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{+2R_1+R_2 \mapsto R_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} = L$$

Equivalent Procedure to find $L$.

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}}$$

$$\boxed{U = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

7. (22 points) (a) (8 points) Write on the space provided **T** if the statement is True or **F** if it is False. No justification needed.

( F ) If $A\mathbf{x} = \mathbf{0}$ has a unique solution, then the nullspace of $A$ is empty.

$\vec{x} = \vec{0}$ is in $Null(A)$, i.e., $Null(A) = \{\vec{0}\}$

( T ) If $\mathbf{u}$ and $\mathbf{v}$ belong to a subspace $W \subset \mathbb{R}^n$, then $5\mathbf{u} + 12\mathbf{v}$ also belongs to $W$.

By property of closure under addition of a Subspace

( T ) If $A$ is a $10 \times 13$ matrix, then the nullspace of $A$ is never $\{\mathbf{0}\}$.

$A\vec{x} = \vec{0}$ ~~is an over determined system~~
has more unknowns than equations. In other words,
$\dim(Null(A)) = nullity(A) = 13 - rank(A) \geq 3$

( F ) A subset of $\mathbb{R}^n$ must be closed under vector addition.

a subset is not necessarily a subspace

( F ) The row space of any matrix equals its column space.

Let $A$ be $m \times n$ matrix
$Row(A) \subset \mathbb{R}^n$, $Col(A) \subset \mathbb{R}^m$

( T ) The dimension of the row space of any matrix is the same as the dimension of its column space.

We know that $rank(A) = rank(A^T)$

( F ) Every matrix has an $LU$ decomposition.

No, when row interchanges are needed to obtain $U$

( F ) A square matrix is invertible if and only if $\det(A) = 0$.

$A$ invertible if and only if $\det(A) \neq 0$

(b) (14 points) Write on the space provided **T** if the statement is True or **F** if it is False. You MUST JUSTIFY your answers. If it is false, give a spacific counterexample, if it is true, prove it is true.

( F ) (5 points) If $A$ and $B$ are two $n \times n$ matrices then $\det(A+B) = \det(A) + \det(B)$.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow A + B = I_2$. Thus $\det(A+B) = \det(I_2) = 1$

But, $\det(A) = 0 = \det(B)$.

Therefore, $1 = \det(A+B) \neq 0 = \det(A) + \det(B)$

( F ) (4 points) If $A$ is a $5 \times 5$ matrix, then $\det(-A) = \det(A)$.

MANY OPTIONS HERE. I will give only 2 of them.

OPTION 1: $\det(-A) = (-1)^5 \det(A) = -\det(A)$

Thus, for $A$ such that $\det(A) \neq 0$

$\det(-A) \neq \det(A)$. ∎

OPTION 2 Let $A = I_5 \Rightarrow \det(A) = 1$,
and $-A = -I_5 \Rightarrow \det(-A) = \det(-I_5) = -1$ ∎

( F ) (5 points) Let $A$ be a square matrix such that $A^2 + A + 3I_n = 0$. Then $A$ is invertible and $A^{-1} = \frac{A^2+A}{3}$

where $0$ is the $\vec{0}$ matrix, $\vec{0}$ is of the same size as $A$

Observe that $A^2 + A + 3I_n = \vec{0} \iff A^2 + A = -3I_n$

$\iff A(A + I_n) = -3I_n$

$\iff A\left(\frac{A+I_n}{-3}\right) = I_n$.

Anagously, $A^2 + A = -3I_n$

$\iff (A+I)A = -3I_n$

$\iff \left(\frac{A+I}{-3}\right)A = I_n$

$\therefore A\left(\frac{A+I}{-3}\right) = I_n = \left(\frac{A+I}{-3}\right)A$. Thus $\boxed{A^{-1} = \frac{A^2+A}{-3}}$