Math 250
Fall 2019
Midterm I
October 10, 2019
Time Limit: 80 Minutes

Name (Print): Answer Key

NetID Blue

Welcome to your Midterm! You have 80 minutes to take this exam, for a total of 105 points. No books, notes, calculators, cellphones or any kind of electronic device are allowed. `Remember that you are not only graded on your final answer, but also on your work. Thus, unless stated otherwise, you MUST justify your answers.`

This exam contains 14 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

If you need to continue your work on a different page, clearly indicate so, or else your work will be discarded. Do not write in the table that is on the second page of this midterm. There is an extra sheet at the end of this test that you can use if you need more space. If the extra page is not enough extra-space for you, please, let me know so I can give you another paper sheet.

Academic Honesty Statement: I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to the academic dishonesty policy and procedures of the Academic Integrity of Rutgers University Statement.

Signature: ________________________

1. (20 points) Write on the space provided **T** if the statement is True or **F** if it is False. NOTE: In this question, you do NOT have to justify any answers! Also, do not spend too much time on each question! (If you do not know the answer MOVE ON! There are more questions to answer in the test)

(T) If A is a 3×3 matrix such that the system $Ax = 0$ has only the trivial solution, then the equation $Ax = b$ is consistent for every b in $\mathbb{R}^3$.

3 pivot columns

(T) If A is a 2×3 matrix, then $Ax = 0$ always has a nonzero solution.

(T) If the **augmented** matrix of the system $Ax = b$ has a pivot in the last column, then the system $Ax = b$ has no solution.

(F) If A and B are any 2×2 matrices, then $BA = AB$.

(T) If A is a 3×3 matrix with two pivot positions, then $Ax = 0$ has a nontrivial solution.

(T) If A is a 2×3 matrix, then $Ax = 0$ always has a nonzero solution.

(F) If a system of linear equations has more variables than equations, then it must have infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(F) There are systems of linear equations with only two solutions.

(T) The equation $Ax = 0$ always has either one or infinitely many solutions.

(F) If A, B, C are square matrices with $AB = BC$, then $B = C$

3. (10 points) A matrix A and a row echelon form of A (we call it U) are given here

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & -1 \\ -2 & -4 & 3 & -3 & 0 \\ 1 & 2 & -3 & 3 & 3 \\ 1 & 2 & -2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Write the general solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$ in vector form.

$$U \xrightarrow{R_1 \leftarrow R_2 + R_1} \begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$x_1 = -2x_2 + 3x_5$$

$$x_3 = x_4 + 2x_5$$

x_2, x_4, x_5 free

$$\Rightarrow \vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

5. (10 points) Find all the values of h and k (if any) for which $A\mathbf{x} = \mathbf{b}$ has
   (a) (3 points) No solutions
   (b) (4 points) Exactly one solution
   (c) (3 points) Infinitely many solutions

$$A = \begin{bmatrix} 1 & -2 \\ 2 & h \\ 3 & -6 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ k \\ -3 \end{bmatrix}$$

$$[A \mid \vec{b}] \xrightarrow[\substack{-2R_1+R_2 \mapsto R_2 \\ -3R_1+R_3 \mapsto R_3 \\ R_1+R_4 \mapsto R_4}]{} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 4+h & 0 \\ 0 & 0 & k-9 \\ 0 & 0 & 0 \end{array}\right]$$

Thus

(a) If $k \neq 9$ and any h value (a row of the form $[0\ 0 \mid d]$ with $d \neq 0$

(b) $k=9$ and $h \neq -4$ (two pivot rows = rank(A))

(c) $k=9$ and $h=-4$ (only one pivot row) < rank(A))

7. (10 points) Find the rank (7 pts) and nullity (3 points) of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 0 & -4 & 1 \end{bmatrix}$$

$\textcircled{\#}$ = pivot position

$$A \xrightarrow[\substack{R_2 - R_1 \mapsto R_2 \\ R_3 - 2R_1 \mapsto R_3}]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & -2 & -6 & -1 \end{bmatrix} \xrightarrow[2R_2 + R_3 \mapsto R_3]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Echelon form of A

From here you can conclude that rank(A) = 3 (3 pivot positions means 3 pivot columns) (# non-zero rows)

By definition:

$$\text{Nullity}(A) = \underset{\text{of } A}{\text{number columns}} - \text{rank}(A)$$

$$\Rightarrow \boxed{\begin{array}{l} \text{Nullity}(A) = 1 \\ \text{Rank}(A) = 3 \end{array}}$$

(Finding the RREF was not necessary but it is not wrong either)

$$\xrightarrow[\substack{-R_3 + R_2 \mapsto R_2 \\ -R_3 + R_1 \mapsto R_1}]{} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_2 + R_1]{} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

9. (10 points) Find the elementary matrix E such that $EA = B$.

(a) (4 points)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 3 & 2 \\ 2 & -1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 3 & 2 \\ 0 & -5 & -6 & -4 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(b) (3 points)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 3 & 2 \\ 2 & -1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 0 & 4 \\ -1 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) (3 points)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 3 & 2 \\ 2 & -1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & -3 & -2 \\ 2 & -1 & 0 & 4 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. (10 points) A system of linear equations is called overdetermined if it has more equations than variables. Give examples of overdetermined systems that have:

   (Hint: Use matrices in RREF)

   (a) (3 points) No solutions:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \leftarrow$$

Consistency theorem guarantees that the linear system described by this augmented matrix is inconsistent.

   (b) (4 points) Exactly one solution:

$$[R|\vec{0}] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

rank $(R) = 2 = \#$ columns of R

$\Rightarrow$ If the solution exist it must be unique.

This is an homogeneous system

$\Rightarrow$ $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ must be soln of $R\vec{x} = \vec{0}$

Thus $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only soln here.

   (c) (3 points) Infinitely many solutions:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$R\vec{x} = \vec{0}$ is always consistent. (Homogeneous system)

$\uparrow$

R has a zero column $\Rightarrow$ 1 free variable

$\Rightarrow$ infinite solutions to this system.

4. (15 points) Label the following statements as TRUE or FALSE. In this question, you HAVE to justify your answer!!! This means:

   - If the answer is TRUE, you have to explain WHY it is true (possibly by citing or stating a theorem, or part of a theorem)
   - If the answer is FALSE, you have to give a specific COUNTEREXAMPLE. You also have to explain why the counterexample is in fact a counterexample to the statement!

   Let $\mathbf{R}$ be the RREF of an $m \times n$ matrix $\mathbf{A}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{0}, \mathbf{b} \in \mathbb{R}^m$. NOTE: If you need to give a counterexample, give it for $\mathbf{R}$.

(F) If $m = 4$ and $n = 5$, then $\mathbf{Rx} = \mathbf{b}$ has a solution for all $\mathbf{b}$.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \leftarrow$$

By consistency theorem we know that this system is inconsistent.

(F) If $m = 2$ and $n = 3$, then $\mathbf{Rx} = \mathbf{0}$ has a unique solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

zero columns $\Rightarrow$ free variables

$$\text{nullity}(R) = n - \text{rank}(R) = 3 - 1 = 2 \Rightarrow 2 \text{ free variables} \Rightarrow \text{infinite solutions.}$$

(F) If $m = 4$ and $n = 4$, then for every $\mathbf{b}$ the equation $\mathbf{Rx} = \mathbf{b}$ has a solution containing a free variable (parameter).

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & b_4 \end{array}\right]$$

No free variables because $x_1 = b_1$, $x_2 = b_2$, $x_3 = b_3$ and $x_4 = b_4$

6. (10 points) Let

$$\Sigma = \{ \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix} \},$$

determine whether the set Σ is linearly independent or linearly dependent. In case it is linearly dependent, write the zero vector $[0, 0, 0]^T$ explicitly as a non-trivial linear combination of the vector in Σ.

$$\begin{bmatrix} 2 & -1 & 0 \\ 2 & 2 & 6 \\ 3 & 1 & 5 \end{bmatrix} \xrightarrow[R_3 - \frac{3}{2}R_1 \mapsto R_3]{R_2 - R_1 \mapsto R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 6 \\ 0 & 5/2 & 5 \end{bmatrix} \xrightarrow{R_3 - \frac{5}{6}R_2 \mapsto R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

You can stop here to conclude

$$\xrightarrow{\frac{1}{3}R_2 \mapsto R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 + R_1 \mapsto R_1} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1 \mapsto R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \underset{\text{for } A\vec{x} = \vec{0}}{\Rightarrow} \begin{matrix} x_1 + x_3 = 0 \\ x_2 + 2x_3 = 0 \end{matrix} \qquad \vec{x} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Thus, Σ is linearly dependent and:

$$-1 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

NOTE: This problem could be sold by inspection of the set Σ too.

~~Conclusion must be stated:~~

8. (10 points) Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Calculate the following matrix products, if they are defined, or explain why they do not make sense.

(a) (4 points) $B^T A$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) (2 points) BA Not defined

B is 3x2
A is 3x3 $\Rightarrow$ $3 \times 2 \times 3 \times 3$

Must be the same number!

\# columns on B $\neq$ # rows on A

(c) (4 points) CBA

$2 \times 2 \times 3 \times 2 \times 3 \times 3$

Not defined

Value Must coincide (be the same Value)

\# columns on B $\neq$ # rows on A

and/or

\# columns of C $\neq$ # rows of B